Structurally Constrained $H_{\infty}$ Suboptimal Control Design Using an Iterative Linear Matrix Inequality Algorithm Based on a Dual Design Formulation

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Abstract

A general approach for solving structurally constrained $H_{\infty}$ suboptimal control problems is proposed. The structurally constrained problems include static output feedback control, decentralized control, and fixed controllers for different operating conditions. The approach uses a dual design formulation based on an $H_{\infty}$ state-feedback controller parametrization result. The dual design condition is in the form of a biaffine matrix inequality (BMI). To solve for the control gains from the BMI, an iterative algorithm based on the linear matrix inequality (LMI) technique is developed. In this new approach, the control gains are independent of any Riccati equation solutions.

Keywords: Robust Control, Linear Matrix Inequalities, State Feedback, Static Output Feedback, Simultaneous Stabilization.

1 Introduction

Many practical control design problems can be formulated as solving for a constant state-feedback gain with structural constraints. Such problems include static output feedback control design [10], decentralized control design [16], and simultaneous stabilization, that is, the design of fixed controllers for several operating conditions [2].

In this paper we consider the design of controllers with such structural constraints to satisfy an $H_{\infty}$-norm objective function. It can be shown that the design conditions of these structurally constrained problems are in the form of biaffine matrix equations/inequalities (BMEs/BMIs) or quadratic matrix equations/inequalities (QMEs/QMIs). The solution set of these inequalities is non-convex and, in general, difficult to find directly. Frequently, conservative quadratic terms are added to the BMI/QMI so that the BMI/QMI can be recast into a linear matrix inequality (LMI) [1] which is a convex programming problem and can be readily solved using LMI tools such as [6]. However, such approaches would result in sufficient but not necessary conditions, and would disallow many admissible solutions.

In this paper, instead of obtaining the feedback gain for the system directly, we develop an approach which optimizes with respect to a dual system obtained from a controller parametrization result [5]. Although this approach also results in design conditions in the form of a BMI, a free design parameter can be added to the BMI such that the resulting inequality can be solved iteratively as LMIs. The use of LMIs in this particular approach is novel, because previously LMIs have mostly been applied to control designs with no structural constraints [11], [8].

The main advantage of the dual design approach is that the design conditions are less conservative due to two distinct features. The first feature is the addition of the free design parameter, which can be iteratively adjusted to reduce the conservativeness of the added quadratic terms. The second feature is that the structurally constrained control gain is no longer required to be a function of a positive (semi) definite matrix. In an unconstrained centralized control design, the feedback control gain is derived from the positive (semi) definite matrix solved from...
a set of matrix equations or inequalities. The design of controls with structural constraints frequently follows the same design philosophy, namely, the feedback gain is assumed to be a function of a positive (semi) definite matrix solved from a set of the structurally constrained design conditions [7,15,3]. Such an approach may eliminate many admissible solutions.

The organization of the rest of this paper is as follows. Section 2 formulates the structurally constrained H suboptimal design problem. A dual design approach to solve the structurally constrained design problem is developed in Section 3. Section 4 proposes an iterative LMI algorithm to compute the structurally constrained feedback gain. The applications of the technique to static output feedback design, decentralized control design, and simultaneous stabilization are given in Section 5. Design examples are also contained in Sections 5.

2 Problem Formulation

Consider the linear time-invariant generalized plant $G_{SF}(s)$ with the state-space realization

\begin{align}
\dot{x} &= Ax + B_1w + B_2u \\
Z &= C_1x + D_{12}u
\end{align}

where $x \in \mathbb{R}^n$ is the state variable vector, $w \in \mathbb{R}^{nw}$ represents the disturbance and other external signals, $u \in \mathbb{R}^m$ is the controlled input vector, and $z \in \mathbb{R}^p$ is the controlled output vector.

In this paper, we often use the packed-matrix notation to express a dynamic system, for example, $G_{SF}(s)$ is expressed as

\[ G_{SF}(s) \leftrightarrow \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \end{bmatrix} \] (2)

A structurally constrained constant-gain state-feedback controller for $G_{SF}(s)$ has the form

\[ u = F_dx \] (3)

where the structure of $F_d$ depends on the problem to be solved. For example, in a decentralized control problem, the matrix $F_d$ will be block-diagonal.

The following assumptions are made throughout this paper:

(A1) $G_{SF}(s)$ is stabilizable under the control structure (3).

(A2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$.

(A3) $D_{12}$ is of full column rank.

Denote the closed-loop transfer function of $G_{SF}(s)$ with the desired controller (3) by $T_{zw}^d(s)$. Then $T_{zw}^d(s)$ has the realization

\[ T_{zw}^d(s) \leftrightarrow \begin{bmatrix} A + B_2F_d & B_1 \\ C_1 + D_{12}F_d & 0 \end{bmatrix} \] (4)

The structurally constrained $H_\infty$ suboptimal control problem is stated as follows:

Structurally Constrained Problem (SCP):

Find a structurally constrained constant-gain state-feedback control (3) such that

\[ \| T_{zw}^d(s) \|_\infty \leq \gamma \] (5)

where $\gamma > 0$ is a pre-specified constant.

Assumptions (A2) and (A3) of the SCP are readily verifiable. On the other hand, Assumption (A1) may not always be verifiable. However, the proposed control design can still proceed without requiring a stabilizing controller (3). If the algorithm fails to yield a solution, a designer may gain some insights on how to modify the design problem and the structural constraint to obtain an acceptable solution.

3 Dual Design Formulation

Optimizing directly $F_d$ for SCP using $T_{zw}^d(s)$ will result in a QMI design condition. This condition can be into a LMI using additional quadratic terms. However, these terms may be very conservative. To develop a less conservative approach, we use a full-state feedback parametrization result [5] to formulate a dual design problem.

In the dual design problem, the full-state feedback optimal $H_\infty$ suboptimal control gain will appear explicitly. From [9], the full-state feedback $H_\infty$ suboptimal control problem is to find a centralized control law $u = Fx$ such that the closed-loop transfer function $T_{zw}(s)$ from $w$ to $z$ satisfies

\[ \| T_{zw}(s) \|_\infty \leq \gamma \] (6)

for a given $\gamma > 0$. Defining the Hamiltonian matrix
We denote by $X_\infty = \text{Ric}(H_\gamma) \geq 0$ as the positive semi-definite matrix solution to the algebraic Riccati equation

$$A_H^T X + XA_H + X R_\gamma X - Q_\gamma = 0$$

where

$$A_H = A - B_2(D_{12}^T D_{12})^{-1}D_{12}^T C_1$$

$$R_\gamma = \gamma^2 B_1 B_1^T - B_2(D_{12}^T D_{12})^{-1}B_2^T$$

$$Q_\gamma = -C_1^T (I - D_{12}(D_{12}^T D_{12})^{-1}D_{12}^T) C_1$$

The matrix $H_\gamma$ for which $\text{Ric}(H_\gamma)$ is defined is the domain of the Riccati operator and is denoted as $\text{dom}(\text{Ric})$. The details of these properties and the notations can be found in [5].

The dual design problem is stated in the following theorem.

**Theorem 1** Suppose that the generalized plant $G(s)$ (1) satisfies Assumptions (A1) - (A3) and $D_{12}$ has the singular value decomposition $D_{12} = U \Sigma V^T$, where $U$ and $V$ are unitary matrices and $\Sigma$ is a diagonal matrix. Then, for a given $\gamma > 0$, if $H_\gamma \in \text{dom}(\text{Ric})$, $X_\infty = \text{Ric}(H_\gamma) \geq 0$, and the structurally constrained constant feedback gain $F_d$ is such that

$$\left\| \begin{bmatrix} A_{\text{tmp}} + B_2 F_d & B_1 \\ S_u(F_d - F) & 0 \end{bmatrix} \right\|_\infty < \gamma$$

where

$$A_{\text{tmp}} = A + \gamma^{-2} B_1 B_1^T X_\infty$$

$$S_u = \Sigma V^T$$

$$F = -(D_{12}^T D_{12})^{-1}(B_2^T X_\infty + D_{12}^T C_1)$$

then the controller $u = F_d x$ is a stabilizing controller satisfying $\| T_{uv}(s) \|_\infty < \gamma$.

To prove Theorem 1, we first obtain the central-ized full-state feedback solution to (6) and the parametrization of all controllers satisfying (6). Then we establish a stable parameter to obtain $F_d$ with the desired structure.

**Lemma 1** If $(A, B_2)$ is stabilizable and Assumptions (A2)-(A3) are satisfied, then there exists an admissible full-state centralized feedback $u = F x$ for (6) such that $\| T_{uv}(s) \|_\infty < \gamma$ if and only if $H_\gamma \in \text{dom}(\text{Ric})$ and $X_\infty = \text{Ric}(H_\gamma) \geq 0$. Furthermore, all admissible controllers satisfying $\| T_{uv} \|_\infty < \gamma$ can be parametrized as $F(M_{\text{SF}}, Q)$ as shown in Figure 1 where

$$M_{\text{SF}}(s) \leftrightarrow \begin{bmatrix} A_{\text{tmp}} + B_2 F & B_1 \\ 0 & F & S_u^{-1} \end{bmatrix}$$

**Figure 1:** State-Feedback Controller Parametrization

And the stable parameter $Q(s)$ satisfies

$$\| Q(s) W(s) \|_\infty < \gamma$$

where

$$W(s) \leftrightarrow \begin{bmatrix} A_{\text{tmp}} + B_2 F & B_1 \\ S_u(F_d - F) & 0 \end{bmatrix}$$

Lemma 1 is the same as Lemma 4 in [17] except for the incorporation of the input transformation matrix $S_u$. The matrix $S_u$ is not needed in [17] because it is assumed that $D_{12} = [0 I]^T$. Keeping the term $S_u$ in $M_{\text{SF}}(s)$ is important for perserving the structure of $F_d$. The proof of Lemma 1 follows closely the proof of Theorem 1 in [17] for the so-called full-information problem, and therefore, is omitted. To use Lemma 1, we establish a result containing an explicit form of $Q(s) = Q_d(s)$ such that the resulting controller $u = F(M_{\text{SF}}, Q_d)x = F_d x$ will have the desired structure.

**Lemma 2** If $H_\gamma \in \text{dom}(\text{Ric})$ for some $\gamma > 0$, then the set of all constant gain matrices $F_d$ with the desired structure is given by $F_d = F(M_{\text{SF}}, Q_d)$ where

$$Q_d(s) \leftrightarrow \begin{bmatrix} A_{\text{tmp}} + B_2 F_d & B_2(F_d - F) \\ S_u(F_d - F) & S_u(F_d - F) \end{bmatrix}$$

**Proof:**

Partition $M_{\text{SF}}(s)$ as

$$M_{\text{SF}}(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix}$$

Using linear fractional transformations, the desired control is

$$F_d = F(M_{\text{SF}}, Q_d) = M_{11} + M_{12} Q_d (I - M_{22} Q_d)^{-1} M_{21}$$

From (21), $Q_d(s)$ is readily solved as

$$Q_d = (I + M_k M_{22})^{-1} M_k$$

(22)
where
\[
M_b = M_1^{-1}(F_d - M_1)M_2^{-1}
\]  
(23)

The state-space realization of \(Q_d(s)\) in (19) can be shown as a minimal realization of (22) [13].

We note the presence of \(S_u\) in the parameter \(Q_d(s)\) (19). Without \(S_u\), even though \(F_d\) has the desired structure, the resulting control \(P(M_{SP},Q_d)\) will not have the required structure. For systems in the standard form, that is, \(D_{12} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\), we have \(S_u = I\). In this case, \(S_u\) can be dropped from \(Q_d\).

**Proof of Theorem 1:**

First we note that the stabilizability of \((A,B_2)\) is guaranteed by Assumption (A1). Combining the results of Lemmas 1 and 2, if there exists a stable \(Q_d(s)\) (19) such that

\[
\| Q_d(s) W(s) \|_\infty < \gamma
\]  
(24)

then the constrained gain matrix \(F_d\) solves SCP. However, it can be readily shown [13] that the minimal realization of \(Q_d(s) W(s)\) is

\[
\begin{bmatrix}
A_{Imp} + B_2 F_d & B_1 \\
S_u(F_d - F) & 0
\end{bmatrix}
\]  
(25)

which completes the proof of Theorem 1.

### 4 Design Algorithm

From Theorem 1, the solution of SCP (5) reduces to the solution of the dual design problem of finding \(F_d\) with the desired structure to satisfy (12). A direct application of the bounded-real lemma to (12) gives a design condition that is both necessary and sufficient.

**Lemma 3 [15]:** The bound (12) is satisfied if and only if the quadratic matrix inequality (QMI)

\[
A_{F_d}^T M + M A_F - M B_2 B_1^T M + (F_d - F)^T S_u^{-1} S_u(F_d - F) \leq 0
\]  
(26)

admits a solution pair \((M, F_d)\) where \(A_{F_d} = A_{Imp} + B_2 F_d\), \(M \geq 0\) and \(F_d\) is a stabilizing control gain for \(Q_d(s)\).

The inequality (26) is a QMI in the unknowns \(M\) and \(F_d\). In many previous approaches to similar problems [15, 3, 7, 14], \(F_d\) is assumed to be a structurally constrained function of \(M\). Then an iterative solution algorithm is used to solve the resulting Lyapunov or Riccati equations. This approach may reduce significantly the admissible solution set. In addition, the iterative solution algorithm may have poor convergence properties. To avoid these difficulties, we reformulate (26) by introducing a free design parameter. First, we observe that from the Schur complement formula in Appendix A, the QMI (26) in the unknowns \(M\) and \(F_d\) can be put into the form

\[
\begin{bmatrix}
A_{F_d}^T M + M A_F - M B_2 B_1^T M + (F_d - F)^T S_u^{-1} S_u(F_d - F) \\
\gamma^{-1} B_1^T M & -I \\
S_u(F_d - F) & 0 & -I
\end{bmatrix} \leq 0
\]  
(27)

which is a BMI due to the product terms of \(M\) and \(F_d\) in the (1,1) entry of the matrix on the left hand side. This BMI cannot be further simplified, and thus solving for \(M\) and \(F_d\) is a non-convex programming problem which is hard. However, if we rearrange (26) to obtain

\[
A_{F_d}^T M + M A_F - M B_2 S_u^{-1} S_u^{-T} B_1^T M + \gamma^{-2} M B_1 B_1^T M + \phi^T S_u^{-1} S_u \phi \leq 0
\]  
(28)

where \(A_F = A_{Imp} + B_2 F\), and \(\phi = F_d - F + S_u^{-1} S_u^{-T} B_1 M\), then we can introduce a matrix parameter \(X\) having the same dimension as \(M\), to reformulate (26) into a different QMI as stated in the following theorem.

**Theorem 2** If there exist \(M \geq 0\), a constant gain \(F_d\) having the desired structure, and \(X\) such that the QMI

\[
\begin{bmatrix}
\Delta & \gamma^{-1} M B_1 & \phi^T S_u^{-1} S_u \phi \\
\gamma^{-1} B_1^T M & -I & 0 \\
S_u \phi & 0 & -I
\end{bmatrix} \leq 0
\]  
(29)

where \(B_s = B_2 S_u^{-1}, B_{ss} = B_s B_s^T\), and \(\Delta = A_{F_d}^T M + M A_F - X^T B_s X - M B_{ss} X + X^T B_{ss} X\), then \(F_d\) solves the SCP (5) and the dual design problem (12).

**Proof:**

The inequality (28) rewritten using the symbols defined in the statement of the theorem is

\[
A_{F_d}^T M + M A_F - M B_2 B_1^T M + \gamma^{-2} M B_1 B_1^T M + \phi^T S_u^{-1} S_u \phi \leq 0
\]  
(30)

Because of the negative sign in the \(- M B_s B_s^T M\) term, (30) cannot be simplified to an LMI. To accommodate the \(- M B_s B_s^T M\) term, we introduce an additional design variable \(X\) having the same dimension as \(M\). Because \((X - M)^T B_s B_s^T (X - M) \geq 0\) for any \(X\) and \(M\), we obtain

\[
X^T B_s B_s^T M + M B_s B_s^T X - X^T B_s B_s^T X \leq M B_s B_s^T M
\]  
(31)

The equality holds if \(X = M\). Combining (31) and (30), the sufficient condition for the existence of \(F_d\) is

\[
\Delta + \gamma^{-2} M B_1 B_1^T M + \phi^T S_u^{-1} S_u \phi \leq 0
\]  
(32)
From Lemma 5 in Appendix A, (32) is equivalent to (29).

At a first glance, it is not obvious that the QMI (29) is any more useful than the BMI (27). However, if $X$ is fixed in (29), then (29) reduces to an LMI for a given $\gamma$ in the unknowns $F_d$ and $M \succeq 0$. If $X$ is admissible, then the LMI problem is convex and can be solved using existing LMI solvers \[6\]. A comprehensive treatment and an extensive list of references of the LMI techniques can be found in [1]. Note that in the LMI approach, $F_d$ is no longer required to be a function of $M$.

The most important step in this LMI approach is the selection of $X$. Ideally we like to select $X = M$, so that the QMI (29) is both necessary and sufficient. However, it is unlikely that a designer will have apriori an admissible $X$. As a result, an iterative algorithm is proposed to find $X$. The iteration starts from an unconstrained problem for which $X$ can be readily found from the centralized solution. Then $X$ is updated by gradually enforcing the structural constraints. Using these ideas, we propose the following iterative LMI algorithm to solve for $X$, $M$, and $F_d$ for a given $\gamma$.

**Iterative LMI (ILMI) Algorithm**

S1: Select $\gamma > 0$ and compute the centralized solutions $X_{\infty}$ and $F$. Separate the gains of $F$ conforming to the structure of $F_d$ and put them in $F_{\text{de}}$. Write $F$ as

$$F = F_{\text{de}} + F_{\text{com}}$$

where $F_{\text{com}}$ contains all the other gains.

S2: Set up the sequence $\epsilon_k$ as

$$\epsilon_k = 1 - \frac{k}{N}, \quad k = 1, 2, \cdots, N$$

S3: Set $k = 1$ and $X^{(1)} = X_{\infty}$.

S4: Solve the following optimization problem OP for $M^{(k)}$ and $F_{d}^{(k)}$.

**OP:** Minimize $\text{trace}(M^{(k)})$ subject to the following LMI constraints

$$\begin{bmatrix} \Delta^{(k)} & \gamma^{-1}M^{(k)}B_1 & \Phi^{(k)}T \Phi^{(k)} \\ \gamma^{-1}B_1^TM^{(k)} & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \leq 0$$

$$M^{(k)} = (\Phi^{(k)}T)S_a^{-1}S_0 \Phi^{(k)} \geq 0$$

where

$$\Delta^{(k)} = A_T^{(k)}M^{(k)} + M^{(k)}A_F - (X^{(k)})TB_{ss}^TM^{(k)}$$

$$- M^{(k)}B_{ss}X^{(k)} + (X^{(k)})TB_{ss}X^{(k)}$$

$$\Phi^{(k)} = F_{d}^{(k)} + \epsilon_kF_{\text{com}} - F + S_0^{-1}B_T^{(k)}M^{(k)}$$

S5: If $\|X^{(k)} - M^{(k)}\| < \delta$, a pre-determined tolerance, go to Step S6. Else set $X^{(k+1)} = M^{(k)}$ and go to Step S4.

S6: If $k = N$, stop. Else set $X^{(k+1)} = M^{(k)}$ and increment $k$ to $k + 1$. Go to Step S4.

Note that in the ILMI algorithm, we skip the $k=0$ iteration because the centralized solutions $F_d^{(0)} = F$ and $M^{(0)} = X_{\infty}$ are the solution to the OP problem with $k = 0$. As $k$ increases, the algorithm attempts to lower the feedback contribution from $F_{\text{com}}$. If the OP problem is successfully solved for $\epsilon = N = 0$, then $F_d = F_d^{(N)}$ solves the SCP problem. This iterative technique is akin to the homotopy method [12], except that $\epsilon$ is taken as a discrete sequence rather than a continuous variable.

There are two iterative loops in the ILMI algorithm. The outer loop uses the $\epsilon - k$ sequence, and the inner loop is contained in the OP problem. The outer loop may fail if the selected $\gamma$ is too small, or the algorithm may converge to a infeasible region. However, the inner loop of the OP problem consisting of a matrix-trace optimization with LMI constraints always converges if the initial $X^{(k)}$ selected is admissible.

**Lemma 4** For a given $\gamma$, $\epsilon_k$ and $F_{\text{com}}$, if the OP problem yields the solution $M^{(k)}$ and $F_{d}^{(k)}$ for a given $X^{(k)}$, then the OP problem is solvable for $X^{(k)} = M^{(k)}$ and yields a lower trace($M^{(k)}$). Thus the OP problem converges if the initial $X^{(k)}$ is admissible.

**Proof:**

We rewrite (35) as

$$A_T^{(k)}M^{(k)} + M^{(k)}A_F - (X^{(k)})TB_{ss}M^{(k)}$$

$$- M^{(k)}B_{ss}X^{(k)} + (X^{(k)})TB_{ss}X^{(k)}$$

$$+ \gamma^{-2}M^{(k)}B_1B_1^TM^{(k)}$$

$$+ (\Phi^{(k)})TS_a^{-1}S_0\Phi^{(k)} \leq 0$$

which can be further expressed as

$$A_T^{(k)}M^{(k)} + M^{(k)}A_F - (X^{(k)})TB_{ss}M^{(k)}$$

$$- M^{(k)}B_{ss}X^{(k)} + (X^{(k)})TB_{ss}X^{(k)}$$

$$+ \gamma^{-2}M^{(k)}B_1B_1^TM^{(k)}$$

$$+ (\Phi^{(k)})TS_a^{-1}S_0\Phi^{(k)} \leq 0$$

(40)

If for a given $X^{(k)}$, the OP problem yields the solution $M^{(k)}$ and $F_{d}^{(k)}$, then $M^{(k)}$ and $F_{d}^{(k)}$ are admissible solutions for (40) when $X^{(k)}$ is set to $M^{(k)}$. Furthermore, the OP problem will yield an $M^{(k)}$ with a smaller trace. Because $M^{(k)} \succeq 0$ is bounded below, the OP problem converges.

It is important that we are not solving just an LMI
feasibility problem for the OP problem. When the LMI s (35) and (36) were solved using the LMI tools without the optimization of trace(M(\(k\))), we found that M(\(k\)) close to the boundary of the admissible solution set were selected. Letting X(\(k+1\)) = M(\(k\)) will result in an infeasible problem for the k+1 iteration. This is due to the fact that proceeding to a larger \(k\), the solution set gets to be smaller. However, by minimizing trace(M(\(k\))), the solution M(\(k\)) is an interior point of the admissible solution set, which will still be an admissible solution for a smaller \(k\).

If a stabilizing control \(u = F_o d\) is already known but \(\|F_d(\omega)\|_{\infty} = \gamma_o\) is greater than the desired \(\gamma\), we can solve for \(F_o\) from (26) with \(F_d = F_o d\) and \(\gamma = \gamma_o\). Then we can skip outer-loop iteration and directly solve the OP problem for a smaller \(\gamma\) with \(X(1) = M_o\) and \(F_{con} = 0\).

We note that in Step S2, we use a sequence \(\omega \in k\) with a fixed stepsize. The fixed stepsize is not required by the ILMI algorithm. It is straightforward to implement the ILMI algorithm with variable step sizes for the sequence \(\omega \in k\).

The ILMI algorithm has been applied successfully to a variety of problems, some of which are illustrated in Section 5. It has solved many problems for which a Riccati or Lyapunov solution approach has failed.

### 5 Structurally Constrained \(H_\infty\) Suboptimal Control Problems

In this section, we will cast several structurally constrained \(H_\infty\) suboptimal control problems in the SCP formulation (5).

#### 5.1 State-Feedback Decentralized Control Problems

Suppose that the generalized plant \(G_{SF}(s)\) consists of two interconnected subsystems with the state \(x_i \in R^{n_i}\), the input \(u_i \in R^{m_i}\), and the output \(y_i \in R^{p_i}\), such that \(x = [x_1, x_2]^T\), \(u = [u_1, u_2]^T\) and \(y = [y_1, y_2]^T = [x_1, x_2]^T\). It is desired to control the subsystem decentrally, that is \(y_i\) is the only measured variable available to \(u_i\) for feedback control.

Thus a state-feedback decentralized controller for \(G_{SF}\) has the structure \(u_i = F_1 x_1\) and \(u_2 = F_2 x_2\), that is,

\[
F = \begin{bmatrix}
F_1 & 0 \\
0 & F_2
\end{bmatrix}
\]

The closed-loop transfer function of \(G_{SF}\) with the decentralized controller (41), denoted by \(T_{zw}^{dec}\), has the realization

\[
T_{zw}^{dec}(s) = \begin{bmatrix}
A + B_2 F_{dec} C_1 + D_{12} F_{dec} B_1 \\
0
\end{bmatrix}
\]

Then, the decentralized control problem of finding \(F_{dec}\) such that

\[
\|T_{zw}^{dec}\|_{\infty} < \gamma
\]

can be solved using the dual design formulation and the ILMI Algorithm by setting \(F_d = F_{dec}\) (41).

We note that the design for two-channel systems is considered in this paper. However, the results here can readily be generalized to systems with more than two channels.

#### 5.2 \(H_\infty\) Static Output Feedback Suboptimal Control Problem

Consider the linear time-invariant generalized plant \(G_{OF}(s)\) with the state-space realization

\[
\dot{x} = Ax + B_1 w + B_2 u \tag{44a}
\]
\[
z = C_1 x + D_{12} u \tag{44b}
\]
\[
y = C_2 x \tag{44c}
\]

where \(C_2\) is of full row rank.

The \(H_\infty\) static output feedback control problem is to find a control law

\[
u = Ly = LC_2 x \tag{45}
\]

such that the transfer function of the closed-loop system from \(w\) to \(z\),

\[
T_{zw}^{of} \leftrightarrow \begin{bmatrix}
A + B_2 C_2 & B_1 \\
C_1 + D_{12} LC_2 & 0
\end{bmatrix}
\]

is stable and

\[
\|T_{zw}^{of}\|_{\infty} < \gamma
\]

Since \(C_2\) is of full rank, without loss of generality, we assume \(C_2 = [I \ 0]\). This can be achieved by choosing a similarity transformation

\[
T = \begin{bmatrix}
C_2 \\
C_{2,\perp}
\end{bmatrix}^{-1}
\]

where \(C_{2,\perp}\) is the null space of \(C_2\).
where $C_{21}$ is chosen such that $\begin{bmatrix} C_2 \\ C_{21} \end{bmatrix}$ is invertible.

Then an equivalent system is
\[
G_{OF}(s) \leftrightarrow \begin{bmatrix} \hat{A} & B_1 & B_2 \\ \hat{C}_1 & 0 & D_{12} \\ \hat{C}_2 & 0 & 0 \end{bmatrix}
= \begin{bmatrix} T^{-1}AT & T^{-1}B_1 & T^{-1}B_2 \\ \hat{C}_1T & 0 & D_{12} \\ \hat{C}_2T & 0 & 0 \end{bmatrix}
\]
(49)

where
\[
\hat{C}_2 = C_2T = \begin{bmatrix} I & 0 \end{bmatrix}
\]
(50)

Letting
\[
F_d = L\hat{C}_2 = L \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} L \end{bmatrix}
\]
(51)

a suboptimal solution to (47) can be obtained using the ILMI algorithm similar to solving the decentralized feedback gain $F_d = F_{dec}$ except that $F_d$ is restricted to the form (51) instead of block diagonal.

5.3 Decentralized Static Output Feedback Control Problems

Given the generalized output feedback system $G_{OF}(s)$ (44), the $H_\infty$ decentralized static output feedback control is to find a static output feedback gain $L$ with the structure
\[
L = L_d = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}
\]
(52)

such that the transfer function of the closed-loop system from $w$ to $z$

\[
T_{zw}^{dof} \leftrightarrow \begin{bmatrix} A + B_2L_dC_2 & B_1 \\ C_1 + D_{12}L_dC_2 & 0 \end{bmatrix}
\]
(53)

is stable and satisfies
\[
\| T_{zw}^{dof} \|_\infty < \gamma
\]
(54)

Following the development in Sections 5.1 and 5.2, the decentralized static output feedback control is to find a feedback gain $F_d$ with the structural constraint
\[
F_d = L_dC_2 = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \end{bmatrix}
\]
(55)

Again, this problem can be solved using the ILMI algorithm.

5.4 Simultaneous State Feedback Control Problems

In simultaneous state-feedback $H_\infty$ suboptimal control problems, the design objective is to find a constant state feedback gain $F_{sm}$ to stabilize a system with two or more models, each corresponding to a different operating condition, and achieve certain performance and robustness requirements. For illustrative purpose, we assume only two models. The generalized systems are represented as the packed-matrix form
\[
G_{ISM} \leftrightarrow \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & 0 & D_{12i} \\ I & 0 & 0 \end{bmatrix}, \quad i = 1, 2
\]
(56)

The simultaneous state-feedback $H_\infty$ suboptimal control problem is defined as the design of a constant state feedback gain $u = F_{sm}x$ such that all the transfer functions of the closed-loop system from $w$ to $z$, denoted as $T_{zw}^{sm}$

\[
T_{zw}^{sm} \leftrightarrow \begin{bmatrix} A_i + B_{2i}F_{sm} \\ C_{1i} + D_{12i}F_{sm} \end{bmatrix}
\]
(57)

are stable and
\[
J_{\infty}^{sm} = \max \{ \| T_{zw}^{sm} \|_\infty, \| T_{zw}^{sm} \|_\infty \} < \gamma
\]
(58)

The problem can be formulated as a special decentralized state feedback control problem. Construct the augmented generalized system $G_{SM}$ with
\[
G_{SM} = \begin{bmatrix} G_{ISM} & 0 \\ 0 & G_{2SM} \end{bmatrix}
\]
(59)

Then, the goal is to find a decentralized state feedback gain $F_d$ with the structure
\[
F_d = \begin{bmatrix} F_{sm} & 0 \\ 0 & F_{sm} \end{bmatrix}
\]
(60)

such that the closed-loop system from $w$ to $z$, denote as $T_{zw}^{sm}$, is stable and
\[
\| T_{zw}^{sm} \|_\infty < \gamma
\]
(61)

With this formulation, the simultaneous stabilizing feedback gain can be solved using the ILMI algorithm.

5.5 Simultaneous Output Feedback Control Problems

Consider the generalized systems $G_{OSM}(s)$ with the state space realization in packed-matrix form
\[
G_{OSM} \leftrightarrow \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & 0 & D_{12i} \\ C_{2i} & 0 & 0 \end{bmatrix}, \quad i = 1, 2
\]
(62)

where $C_{2i}$ is of full row rank.

The simultaneous $H_\infty$ output feedback suboptimal control problem is to find a control law
\[
u = L_{sm}y = L_{sm}C_2x_i
\]
(63)

such that all the transfer functions of the closed-loop system from $w$ to $z$, denoted as $T_{zw}^{sm}$

\[
T_{zw}^{sm} \leftrightarrow \begin{bmatrix} A_i + B_{2i}L_{sm}C_{2i} & B_{1i} \\ C_{1i} + D_{12i}L_{sm}C_{2i} & 0 \end{bmatrix}
\]
(64)

are stable and
\[ J_{\infty}^{\text{am}} = \max \{ \| T_{1w}^{\text{am}} \|_{\infty}, \| T_{2w}^{\text{am}} \|_{\infty} \} < \gamma \] (65)

Since \( C_2 \) is of full row rank, without loss of generality, we again assume \( C_2 = [I \ 0] \). Following the procedure for the simultaneous state feedback problems, we construct the augmented system

\[ G_{\text{OSM}} = \begin{bmatrix} G_{1\text{OSM}} & 0 \\ 0 & G_{2\text{OSM}} \end{bmatrix} \] (66)

Then the goal is to find a decentralized feedback gain \( F_d \) having the structure

\[ F_d = \begin{bmatrix} L_{\text{am}} & 0 & 0 & 0 \\ 0 & 0 & L_{\text{am}} & 0 \end{bmatrix} \] (67)

for the augmented system \( G_{\text{OSM}} \) such that the closed-loop system from \( w \) to \( z \), denoted by \( T_{2w}^{\text{am}} \), is stable and

\[ \| T_{2w}^{\text{am}} \|_{\infty} < \gamma \] (68)

The ILMI algorithm can be used to solve for the feedback gain \( L_{\text{am}} \).

### 5.6 Design Examples

In this section, the benchmark problem in [4] of controlling a system of two interconnected inverted pendulums in cascade is investigated. From this system, we generate four problems with increasing design difficulties to demonstrate the proposed design approach.

**Example 1: Decentralized Control.**

The inverted pendulum model expressed in (1) has the state-space matrices [4]

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
9.80 & 0 & -9.80 & 0 \\
0 & 0 & 0 & 1 \\
-9.80 & 0 & 2.94 & 0
\end{bmatrix},
\]

\[B_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, B_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

\[C_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, C_{12} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]

The first two states are the angular displacement and the velocity of the first pendulum, and the last two states are those for the second pendulum. This system, as pointed out by Davison [4], is 'highly' unstable and difficult to control decentrally. We use a decentralized control design on the first two states, and \( u_2 \) utilizing the first two states, and \( w_3 \) the second two states. A decentralized feedback can be readily obtained by using the proposed ILMI algorithm. For \( \gamma = 10 \), the centralized state feedback gain (15) is

\[
F = \begin{bmatrix}
-3.0719 & -0.9018 & 2.4183 & 0.3685 \\
6.6010 & 1.8914 & -5.2872 & -0.8774
\end{bmatrix}
\]

and the decentralized feedback gain \( F_d \) obtained from dual design problem (12) is

\[
F_d = \begin{bmatrix}
-17.088 & -4.4031 & 0 & 0 \\
0 & 0 & -5.7216 & -0.1549
\end{bmatrix}
\]

(72)

(73)

With these feedback gains, the \( H_{\infty} \) norm of the dual design problem (12) is 8.814 and the \( H_{\infty} \) norm of the closed-loop system is found to be 8.885. We remark that \( F_d \) (73) achieves a lower \( H_{\infty} \) norm of the closed-loop system than the control found in [7] which achieves an \( H_{\infty} \) norm of about 27.

**Example 2: Output Feedback Control.**

We now restrict the output of the pendulum system to be

\[
C_2 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

(74)

Since the output matrix \( C_2 \) is not in the form \([I \ 0]\), we first select a similarity transformation

\[
T = \begin{bmatrix}
0.5 & 0 & -0.7071 & 0 \\
0.5 & 0.7071 & 0 & 0 \\
0 & 0.5 & 0 & -0.7071 \\
0 & 0.5 & 0.7071 & 0
\end{bmatrix}
\]

(75)

Then

\[
\hat{A} = \begin{bmatrix}
5.4000 & -4.9000 & -6.2225 & 6.9296 \\
-4.9000 & 1.9700 & 6.9296 & -1.3718 \\
3.1113 & -3.4648 & -5.4000 & 4.9000 \\
-3.4648 & 0.6859 & 4.9000 & -1.9700
\end{bmatrix}
\]

(76)

\[
\hat{B}_1 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
-0.7071 & 0.7071 & 0 & 0 \\
0 & 0 & 0.7071 & 0.7071
\end{bmatrix}, \hat{B}_2 = \begin{bmatrix}
1 & -2 \\
-2 & 5 \\
0.7071 & -1.4142 \\
-1.4142 & 3.5355
\end{bmatrix}
\]

(77)

\[
\hat{C}_1 = \begin{bmatrix}
0.5 & 0 & -0.7071 & 0 \\
0 & 0.5 & 0 & -0.7071 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \hat{C}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(78)

(79)

(80)
For $\gamma = 22$, the ILMI algorithm yields the output feedback gain

$$L = \begin{bmatrix} -1.5576 & 1.0097 \\ 5.1598 & -4.4122 \end{bmatrix}$$

(81)

Using this output feedback gain, the $H_\infty$-norm of the dual design problem is 19.337 and the $H_\infty$-norm of the closed-loop system is 19.419.

In a decentralized output feedback setting with $u_1$ utilizing only $y_1$ and $u_2$ utilizing only $y_2$, we obtained, for $\gamma = 25$, the decentralized control

$$L_d = \begin{bmatrix} -113.05 & 0 \\ 0 & -1.3202 \end{bmatrix}$$

(82)

For this feedback gain, the $H_\infty$-norm of the dual design problem is 20.110 and the $H_\infty$-norm of the closed-loop system is 20.203.

**Example 3:** Simultaneous Stabilization - state feedback.

In addition to the model in Example 1, we assume that the pendulum system also operates at a weakened interconnection state. Thus we have two models of the system:

- Generalized system 1:
  
  $$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.80 & 0 & -9.80 & 0 \\ 0 & 0 & 0 & 1 \\ -9.80 & 0 & 2.94 & 0 \end{bmatrix}$$

  (83)

  $$B_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  $$B_{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

  $$B_{22} = B_{21}, \quad C_{12} = C_{11}, \quad D_{122} = D_{121}$$

  (87)

- Generalized system 2:
  
  $$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.80 & 0 & -4.90 & 0 \\ 0 & 0 & 0 & 1 \\ -4.90 & 0 & 2.94 & 0 \end{bmatrix}, \quad B_{12} = B_{11}$$

  (86)

From the ILMI algorithm, the simultaneous stabilizing feedback gain $F_{sm}$ is found to be

$$F_{sm} = \begin{bmatrix} -11.774 & -3.7859 & 4.5831 & 0.3777 \\ 15.580 & 5.1441 & -7.4765 & -0.9213 \end{bmatrix}$$

(90)

The $H_\infty$-norm of the closed-loop systems subject to the three different feedback gains (88), (89), and (90) are summarized in Table 1.

<table>
<thead>
<tr>
<th>System</th>
<th>$F_{c1}$</th>
<th>$F_{c2}$</th>
<th>$F_{sm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{sm}$ &amp; 2.2640 &amp; 10.952  &amp; 4.8850</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{sm}$ &amp; unstable &amp; 3.8438 &amp; 4.8928</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{sm}$ &amp; - &amp; 10.952 &amp; 4.8928</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 4:** (Simultaneous stabilization - output feedback)

In this example, the two generalized models in Example 3 will be used. We assume that the output matrix for both models are

$$C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(91)

For $\gamma = 20$, the output feedback gains for system 1 ($L_1$) and system 2 ($L_2$) are

$$L_1 = \begin{bmatrix} -1.4711 & 0.9423 \\ 5.1590 & -4.4706 \end{bmatrix}$$

(92)

$$L_2 = \begin{bmatrix} -15.989 & 1.1741 \\ 19.119 & -3.0208 \end{bmatrix}$$

(93)

From the ILMI algorithm, the simultaneous stabilizing output feedback gain $L_{sm}$ is found to be

$$L_{sm} = \begin{bmatrix} -427.80 & 139.03 \\ 193.94 & -65.706 \end{bmatrix}$$

(94)

The $H_\infty$-norm of the closed-loop systems subject to the three different output feedback gains (92), (93), and (94) are summarized in Table 2.

<table>
<thead>
<tr>
<th>System</th>
<th>$F_{c1}$</th>
<th>$F_{c2}$</th>
<th>$F_{sm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{sm}$ &amp; 2.2640 &amp; 10.952  &amp; 4.8850</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{sm}$ &amp; unstable &amp; 3.8438 &amp; 4.8928</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{sm}$ &amp; - &amp; 10.952 &amp; 4.8928</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**6 Conclusions**

An iterative LMI algorithm has been proposed to obtain the numerical solution for the structurally constrained state-feedback $H_\infty$ suboptimal control problems. The technique is based on optimizing with respect to a dual design problem, rather than optimizing directly the closed-loop system. We have shown that the dual design formulation results in design conditions in the form of BMI that is suitable for the development of an iterative LMI solution algorithm. In the iterative LMI algorithm, we introduce a coupling parameter which we adjust...
to achieve the desired control structure. The structurally constrained state feedback gain obtained from the iterative LMI algorithm does not depend on the positive definite Riccati solution from the design conditions. Therefore, the solution region is less restricted than that from the traditional design. As a result, an advantage of our approach is that it will provide superior controllers.

The dual design formulation is applicable to $H_\infty$ suboptimal control problems which can be formulated as a state-feedback control problem with structural constraints on the feedback gains. These problems include decentralized control, static output feedback, and simultaneous optimization problems. The structurally constrained state-feedback gains are obtained using the proposed iterative LMI solution algorithm.

<table>
<thead>
<tr>
<th>system</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_{\inf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{\text{train}}$</td>
<td>18.516</td>
<td>111.78</td>
<td>19.985</td>
</tr>
<tr>
<td>$T_{\text{norm}}$</td>
<td>unstable</td>
<td>6.5510</td>
<td>9.3376</td>
</tr>
</tbody>
</table>

Table 2: $H_\infty$-norm of the closed-loop systems subject to output feedback gains $L_1$, $L_2$, and $L_{\inf}$

**Appendix A**

**Lemma 5** (Schur Complement Formula) [1] For any matrices $\Phi_{11}$, $\Phi_{12}$, and $\Phi_{22}$ where $\Phi_{11}$ and $\Phi_{22}$ are symmetric, the following statements are equivalent:

(i) \[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
\Phi_{12}^T & \Phi_{22}
\end{bmatrix} \geq 0
\]

(ii) $\Phi_{22} \geq 0$, $\Phi_{11} \geq \Phi_{12} \Phi_{22} \Phi_{12}^T$, $\Phi_{12}(I - \Phi_{22} \Phi_{12}^T) = 0$

where $\Phi_{22}$ denotes the Moore-Penrose inverse of $\Phi_{22}$.

**Lemma 6** (Positive Real Lemma) [1] Given the transfer function $H(s) = C(sI-A)^{-1}B+D$ such that all eigenvalues of $A$ have negative real part, $(A,B)$ is controllable and $D + DT^T > 0$, then the following statements are equivalent:

(i) The transfer function $H(s)$ is positive-real, that is,

\[
H(s) + H(s)^* \geq 0
\]

for all $s$ with real part of $s \geq 0$.

(ii) The LMI

\[
\begin{bmatrix}
A^T P + PA & PB - CT \\
BT^T - C & -(D + DT^T)
\end{bmatrix} \preceq 0
\]

in the variable $P = P^T > 0$ is feasible.

**Reference**


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