A Closed Form Solution for the Pull-in Voltage of the Micro Bridge

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Abstract

This paper derives a closed form solution with fringing filed effects for the pull-in voltages of the micro fixed-fixed beam subjected to electrostatic loads and initial stress. The closed form solution is derived based on the Euler’s beam theory and the energy method. The accuracy of the present closed form solution is verified through comparing with the experimentally measured data of the published literatures. The error of the present closed form solution is within 1% compared to the measured data. The present closed form solution is more accurate than the past works and is very simple and highly accurate for implementation in the design of MEMS.

Key Words: Electrostatic, Fringing Field, MEMS, Pull-in Voltage

1. Introduction

The electrostatic principle is a very commonly used principle on sensing or actuating devices in MEMS. It can also be used to extract the material parameters of thin films [1,2]. The nonlinear electrostatic force gives rise to the well-known pull-in phenomenon that causes beams or diaphragms to collapse on the ground if the applied voltage exceeds a certain limit namely the pull-in voltage. Determination of the pull-in voltage is critical in the design of MEMS devices. Accurate determination of pull-in voltage is very challenging in virtue of the electromechanical coupling effect and the nonlinearity of electrostatic force [3,4]. Effects such as the fringing field and the initial residual stress further complicate the modeling.

The finite element method (FEM) is often used for modeling the pull-in phenomenon and has been implemented in various commercial MEMS simulation software. FEM has the disadvantages of un-explicit physical meaning and requiring massive numerical calculations. On the contrary, analytical model can provide a better insight into the physical characteristics of devices. A simple lumped model was often used formerly to simulate the electrostatically actuated microstructures as it consists of a single parallel plate capacitor suspended by an ideal linear spring [1,6]. However, the simple lumped model deviates far from the reality in virtue of ignoring the electromechanical coupling and the fringing field, as well as the distributed deformation of structures. There has been numerous works on analytical modeling of the pull-in voltage for beams or diaphragms [1,2,7–10] which develop from the foregoing simple lumped model with the modified capacitance accounting for the fringing field and the equivalent spring constant. Another kind of analytical model is the distributed model based on well-known beam and plate theories [6]. However, such distributed model has to solve the highly nonlinear governing equations. Some literatures [11,12] assume the deflection function of a cantilever beam as a square function of position and obtain that the pull-in occurs as the tip deflection exceeds about half the original tip gap. Some other analytical models [13,14] assume the deflection functions of electrostatically actuated beams as the polynomial functions that satisfy the boundary conditions. Then the analytical approximated solutions for pull-in voltages are obtained via energy method. There still are reduced-order distributed models that expand the nonlinear electrostatic force term as Taylor series and truncate...
the second and higher terms to obtain linearized governing equations [3,4,15]. However, the reduced order models have a disadvantage of missing the pull-in phenomenon in virtue of truncating the second and higher terms of the Taylor series of electrostatic force. Thus, they assume the pull-in tip deflection as one third of the original tip gap for cantilever beams.

This paper derives a closed form solution with friction filed effects for the pull-in voltages of the micro fixed-fixed beam subjected to electrostatic loads and initial residual stress. The closed form solution is derived based on the Euler’s beam theory and the energy method. The accuracy of the present closed form solution is verified through comparing with the experimentally measured data conducted in the published works [1,2].

2. System Energy Expression

Consider an Euler beam, as shown in Figure 1, with initial residual stress \( \sigma_0 \) and constant cross section, the total potential energy \( U \) is given as Eq. (1), where the first term is the mechanical strain energy and the second term is the electrical potential energy. The \( E, v, I, L, h, b, \) and \( w \) represent the Young’s modulus, the Poisson’s ratio, the cross-sectional area moment of inertia, the length, the thickness, and the deflection function respectively. \( V \) is the electrical potential, \( \varepsilon \) and \( g \) represent the permittivity of the dielectric medium and the initial gap between the beam and the ground plane respectively. The fringing fields are considerable and must be taken into account when modeling the electrical potential energy. A technical report [16] summarizes several different formulas for computing the fringing field capacitances between the on-chip interconnect and the ground plane. Among which, the model proposed by Meijs and Fokkema can compute the capacitance much faster than the others do and the error is small enough. The electrical potential energy is derived exactly based on the Meijs and Fokkema’s model. The error of the potential energy model is small than 2% as \( b/g \geq 1 \) and \( 0.1 \leq h/g \leq 4 \), and small than 6% as \( b/g \geq 0.3 \) and \( 4 \leq h/g \leq 10 \).

3. The Closed-form Solution of Pull-in Voltage

We adopt the first natural mode of the fixed-fixed beam \( \phi(x) \) shown in Eq.’s (2) and (3) as a trial deflection shape function since it satisfies the boundary conditions and is very similar to the case of electrostatic attraction forces. Substitute the trial deflection shape function \( \phi(x) \), the deflection function \( w(x) \) can be expressed as \( w(x) = \eta \phi(x) \), where the coefficient \( \eta \) is to be solved. Substituting into the total potential energy expression Eq. (1) yields Eq. (4). The system is in static equilibrium when the first-order derivative of the total potential energy \( U \) with respect to the coefficient \( \eta \) equals zero. At the transition from a stable to an unstable equilibrium, the second order derivatives of the total potential energy with respect to \( \eta \) also equals zero. Equations (5) and (6) are obtained from the first and second derivative of the total potential energy with respect to \( \eta \) respectively. Dividing Eq. (5) by Eq. (6) gives the \( \eta_{PI} \) at pull-in as Eq. (7). Equation (6) gives the pull-in voltage \( V_{PI} \) as Eq. (8), where the parameters \( S \) and \( B \) depend on the geometrical parameters of the micro bridge and are given as Eq.’s (9) and (10).

\[
U = \int_0^L \left[ \frac{\sigma_0 bh}{2} \left( \frac{d^2 w}{dx^2} \right)^2 + \frac{EI}{2} \left( \frac{d^2 w}{dx^2} \right)^2 \right] dx - \int_0^L \varepsilon V^2 \left( \frac{b}{g-w} \right) + 0.77 + 1.06 \left( \frac{b}{g-w} \right)^{0.25} + 1.06 \left( \frac{h}{g-w} \right)^{0.5} \right] dx \tag{1}
\]

\[
\phi(x) = (\cosh \lambda x - \cos \lambda x) - \zeta (\sinh \lambda x - \sin \lambda x), \tag{2}
\]

\[
\zeta = \left[ \cosh (\lambda L) - \cos (\lambda L) \right] / \left[ \sinh (\lambda L) - \sin (\lambda L) \right], \tag{3}
\]

\[
U = \eta^2 \left[ \frac{\sigma_0 bh}{2} \int_0^L \phi^2 dx + \frac{EI}{2} \int_0^L \phi^2 dx \right] - \frac{e V^2}{2} \int_0^L \left( b \left( \frac{g-\eta \phi}{g-\eta \phi} \right) + 0.77 + 1.06 \left( \frac{b}{g-\eta \phi} \right)^{0.25} + 1.06 \left( \frac{h}{g-\eta \phi} \right)^{0.5} \right] dx \tag{4}
\]

\[
\eta \left[ \sigma_0 bh \int_0^L \phi^2 dx + EI \int_0^L \phi^2 dx \right] = \frac{e V^2}{2} \int_0^L \left[ \frac{b \phi}{(g-\eta \phi)^2} + 0.265 h^{0.25} \phi + 0.53 h^{0.5} \phi \right] dx \tag{5}
\]


4. Results and Discussion

The accuracy of the present closed form solution is verified by comparing with the measured data conducted by the published works [1,2]. Three cases are demonstrated, which are the poly-silicon bridges [2] manufactured by the standard MUMP’s process and the mono-crystalline silicon bridges [1] with the cross section in (110) and (100) crystalline plane and manufactured by bulk micromachining. Table 1 lists the material and geometric parameters of the demonstrated samples. Figures 2 to 4 show the results of the foregoing three demonstrated samples. It is shown that the closed form solution agree very well with the measured data conducted by the published works.

Table 1. Material and geometrical parameters of the micro bridges [1,2]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Osterberg [1]*</th>
<th>Gupta [2]*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width, ( b ) (( \mu m ))</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>Thickness, ( h ) (( \mu m ))</td>
<td>2.94</td>
<td>2.10</td>
</tr>
<tr>
<td>Initial gap, ( g ) (( \mu m ))</td>
<td>1.05</td>
<td>2.34</td>
</tr>
<tr>
<td>Young’s modulus, ( E ) (GPa)</td>
<td>168 in (110), 138 in (100)</td>
<td>157</td>
</tr>
<tr>
<td>Initial stress, ( \sigma_0 ) (MPa)</td>
<td>10</td>
<td>-3.8</td>
</tr>
<tr>
<td>Permittivity, ( \varepsilon ) (F/m)</td>
<td>( 8.85 \times 10^{-12} )</td>
<td>( 8.85 \times 10^{-12} )</td>
</tr>
</tbody>
</table>

* Osterberg’s [1] sample is the monocrytalline silicon beam, Gupta’s [2] sample is the polysilicon beam.
5. Conclusion

This paper develops a closed form solution with very high accuracy for the pull-in voltage of micro bridge, which includes the influence of the fringing filed and the initial stress. The error is smaller than 1% compared to the measured data conducted by the published works. The present closed form solution applies not only to the fixed-fixed beam but also to the other boundary conditions if the assumed deflection shape function satisfies the boundary conditions.

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References


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