Robustly Decentralized $H_\infty$ Controller with Variance Constraint Design for Large-Scale Stochastic Uncertain Time-Delay Systems via LMI Approach

Wen-Ben Wu$^1$, Pang-Chia Chen$^2$, Gong Chen$^3$, Koan-Yuh Chang$^4$ and Yeong-Hwa Chang$^5$

$^1$Department of Electrical Engineering, Chin-Min Institute of Technology, Miaoli, Taiwan 351, R.O.C.
$^2$Department of Electrical Engineering, Kao Yuan University, Kaohsiung, Taiwan 821, R.O.C.
$^3$Department of Electrical and Electronic Engineering, National Defense University, Taoyuan, Taiwan 335, R.O.C.
$^4$Department of Electronic Engineering, Chienkuo Technology University, Changhua City, Taiwan 500, R.O.C.
$^5$Department of Electrical Engineering, Chang Gung University, Taoyuan, Taiwan 333, R.O.C.

Abstract

The present paper investigates the problem of robustly decentralized $H_\infty$ state feedback controller with state variance constraint design for a class of stochastic large-scale uncertain time-delay systems. The considered time-delay parameters appear in the interconnections between individual subsystems and uncertainties are allowed to be unstructured but time-varying and norm-bounded. The sufficient conditions of the desired state feedback controller, which satisfies the performance level constraint in terms of $H_\infty$ norm for noise attenuation and upper bound of individual state variance constraint for considering its state energy limitation, are based on the Lyapunov-Krasovskii stability theory and utilizing the decentralized scheme to be derived in terms of linear matrix inequalities (LMIs). The effectiveness of the proposed approach is illustrated by a numerical example.

Key Words: LMI, Large-Scale Uncertain Time-Delay System, Robustly Decentralized Control, Variance Constraint

1. Introduction

A number of disturbances acting on the systems are random in nature, such that the performance analysis must directly address the stochastic aspect of the problem. Indeed, stochastic systems have received much attention since stochastic modeling has become to play an important role in many branches of science and engineering applications. Many fundamental results for stochastic uncertain time-delay systems with no interconnections have been investigated [1–3]. Unfortunately, large-scale systems, consisting of a set of interconnected lower-dimension subsystems, are frequently encountered in the real world, such as power systems, digital communications networks, flexible manufacturing networks, and so on. Owing to the existence of interconnections among subsystems, the controller design of a large-scale system is in general much more difficult than that of individual system. These difficulties motivate the development of decentralized control theory where each subsystem is controlled independently on its locally available information. Because the advantage of this scheme in controller design
is able to reduce complexity and allows the control implementation to be feasible, the problem of decentralized control of large-scale interconnected systems therefore became an attractive topic and many applications have been extensively reported in the literatures [4–6].

On the other hand, delays are generally inherent in many physical systems due to transportation or computation time. Moreover, many unavoidable uncertainties occur due to aging of system elements, physical temperature variation, system linearization, and so forth. Since delay and uncertainty often cause deterioration of system performance and may be a source of instability. Therefore, some significant results including robust stability analysis and decentralized stabilization for uncertain stochastic large-scale time-delay systems have been proposed in [7] and reference therein. Furthermore, $H_\infty$ controller design for uncertain stochastic time-delay systems with no interconnection has been considered in [8] and reference therein. In [9], an output feedback $H_\infty$ controller design of stochastic large-scale time-delay systems is also addressed. However, in the study of stochastic system, it is desired that the state variables of the closed-loop systems are maintained within certain level of root mean square (RMS) values to avoidably avoid resulting high gain. One way is to address the state covariance upper bound condition during controller design [10,11]. Such an application for the stochastic large-scale systems with time-delays has proposed in [12]. So far, however, still very few stabilization results with $H_\infty$ performance and state variance constraints are available for stochastic large-scale uncertain time-delay systems.

This paper deals with the design problem of robustly decentralized $H_\infty$ controller (RDHC) with state variance constraints for the stochastic large-scale uncertain time-delays systems. In other words, the individual subsystem within designed controller can achieve certain control objectives which include the performance level constraint in terms of $H_\infty$ norm for noise attenuation and upper bound of individual state variance constraint for considering its state energy limitation. In general, the addressed objective control problem considers a mix of time- and frequency-domain specifications as presented in [13,14], where all objectives are formulated in terms of a common Lyapunov function and controller design amounts to solve a system of linear matrix inequalities (LMIs). Recently, the multi-objective control problems have been considerably studied in control engineering via LMI approach due to its computational advantage and simplicity in solving the addressed problems [15–17]. Indeed, the controller parameters which satisfy the above LMI conditions can be easily found by various efficient convex optimization algorithms [18,19].

As the controller design presented in this paper, based on the Lyapunov-Krasovskii functional stability theory [20] and utilizing the decentralized scheme and LMI approach [18,19], we investigate the problem of developing individual RDHC with state variance constraint for each stochastic uncertain time-delay subsystem. The considered time-delay parameters appear in the interconnections between individual subsystems and uncertainties are allowed to be unstructured but time-varying and norm-bounded. Eventually, the resulting individual decentralized state feedback controllers can ensure the corresponding overall closed-loop uncertain time-delay systems to achieve the addressed $H_\infty$ performance and state variance constraints.

The remainder of the present paper is organized as follows. In Section 2, the properties of the stochastic large-scale uncertain time-delay systems are introduced and the desired objective performance control problems are formulated. In Section 3, an algorithm for constructing robustly decentralized stabilization in probability with $H_\infty$ performance and state variance constraints is developed by using Lyapunov-Krasovskii functional stability theory, decentralized scheme and LMI approach. The effectiveness of the current work is illustrated by a numerical example in Section 4. Finally, some conclusions are given in Section 5.

### 2. Problem Statement and Formulation

Consider the stochastic large-scale systems with time delays consisting of $N$ interconnected subsystems, each subsystem can be represented as the following dynamic equations:

\[
\begin{align*}
\dot{x}_i(t) &= (A_i + \Delta A_i)x_i(t) + (B_i + \Delta B_i)u_i(t) \\
& \quad + D_i w(t) + \sum_{j=1}^{N} (A_{ij} + \Delta A_{ij})x_j(t - \tau_{ij}) \\
\tau_i(t) &= F_i x_i(t) \\
x_i(\theta) &= \phi(\theta), \quad \forall \theta \in [-\tau_i, 0], \quad \tau_i > 0 \\
i, j = 1, 2, \ldots, N, \quad j \neq i
\end{align*}
\]
Where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $z_i(t) \in \mathbb{R}^{q_i}$, $i = 1, 2, \ldots, N$ are the state variable, control input and controlled output of the $i^{th}$ subsystem, respectively, $w_i(t) \in \mathbb{R}^{p_i}$, $i = 1, 2, \ldots, N$, is the white noise input defined on a filtered probability space $(\Omega, \mathcal{F}, (F_i)_{i=1}^{N}, \mathcal{P})$ which satisfies the following properties:

$$
\begin{align*}
\mathbb{E}[w_i(t)] &= 0, & \mathbb{E}[x_i(0)w_i^T(t)] &= 0, \\
\mathbb{E}[w_i(t)w_i^T(t)] &= I, & \mathbb{E}[w_i(t)w_j^T(t)] &= 0, & i, j = 1, 2, \ldots, N, & i \neq j
\end{align*}
$$

(2)

Furthermore, in equation (1), $\tau_{ij} \geq 0$, $i = 1, 2, \ldots, N$, $j \neq i$, is the time-delay existing in the interconnections, and $\phi(0) \in \bar{C}[-\tau_{ij}, 0]$ is the initial condition, where $\bar{C}[-\tau_{ij}, 0]$ stands for a space of continuous functions defined on $[-\tau_{ij}, 0]$. $A_i$, $B_i$, $D_i$, $F_i$ and $G_{ij}$, $i = 1, 2, \ldots, N$, $j \neq i$, are the system constant matrices with appropriate dimensions, and $A_{ij}$ are interconnection matrices between the $i^{th}$ and $j^{th}$ subsystem. $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta A_{ij}(t)$, $i,j = 1, 2, \ldots, N$, $j \neq i$, are matrices representing system time-varying parameter uncertainties which are assumed to be of the form

$$
\Delta A_i(t) = H_{ai}S_a(t)E_{a} \\
\Delta B_i(t) = H_{bi}S_b(t)E_{b} \\
\Delta A_{ij}(t) = H_{ai}S_{ij}(t)E_{ij}
$$

(3)

where $H_{ai}$, $H_{bi}$, $H_{ai}$, $E_{ai}$, $E_{bi}$, and $E_{ai}$, $i = 1, 2, \ldots, N$, $j \neq i$, are known constant matrices; $S_a(t)$, $S_b(t)$, and $S_{ij}(t)$, $i = 1, 2, \ldots, N$, $j \neq i$, are unknown real time-varying matrices with Lebesgue measurable elements satisfying the following norm bounded conditions:

$$
\begin{align*}
S_a(t)S_a^T(t) &\leq I, & S_b(t)S_b^T(t) &\leq I, \\
S_{ij}(t)S_{ij}^T(t) &\leq I, & \forall t
\end{align*}
$$

(4)

Then, $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta A_{ij}(t)$ are said to be admissible if both conditions (3) and (4) hold. Moreover, assume that the pair $(A_i, B_i)$, $i = 1, 2, \ldots, N$, is completely stabilizable for the $i^{th}$ subsystem and all the state variables are measurable.

When a decentralized state feedback control law

$$
u_i(t) = G_i x_i(t), \quad i = 1, 2, \ldots, N
$$

(5)

is applied to the system (1), the closed-loop system is governed by

$$
\begin{align*}
\dot{x}_i(t) &= (\hat{A}_i + \Delta \hat{A}_i)x_i(t) + \sum_{j=1}^{N}(A_{ij} + \Delta A_{ij})x_j(t - \tau_{ij}) + D_iw_i(t), \\
z_i(t) &= F_i x_i(t), \quad i,j = 1, 2, \ldots, N, \quad i \neq j
\end{align*}
$$

(6)

Where $\hat{A}_i \triangleq (A_i + B_iG_i)$ and $\Delta \hat{A}_i \triangleq (\Delta A_i + \Delta B_iG_i)$, in which $G_i$ is the state feedback gain matrix of the $i^{th}$ subsystem with appropriate dimensions to be determined.

Now, the purpose of the current paper is based on the decentralized scheme to design a decentralized state feedback control law (5) for the $i^{th}$ subsystem such that the overall closed-loop system (6) is robustly stochastically stabilizable in probability and satisfies the following formulated conditions:

**Objective (i):** Constraints on $H_{ai}$ norm

Under the zero initial condition, the desired disturbance attenuation performance level is described as follows [8]:

$$
\sum_{i=1}^{N}\mathbb{E}\left[\left\|z_i(t)\right\|_2^2\right] < \gamma_i \sum_{i=1}^{N}\mathbb{E}\left[\left\|w_i(t)\right\|_2^2\right],
$$

\quad $w_i(t) \in L_2[0, \infty), \quad i = 1, 2, \ldots, N

$$
\left\|z_i(t)\right\|_2 = \left(\int_{0}^{\infty} z_i^2(t)dz(t)dt\right)^{1/2}
$$

(7)

and the performance level upper bound $\gamma_i$ is a positive scalar which can be implemented as a constraint to be met or a parameter to be minimized during the controller construction.

**Objective (ii):** Constraints on state variance upper bound

The individual steady-state variance of the system (6) is expected to satisfy the following constraint:

$$
Var(x_k(t)) \leq \left[\hat{X}_i\right]_{kk} \leq (\sigma_k^2), \quad k = 1, 2, \ldots, n
$$

(9)

where $Var(x_k(t))$ and $\sigma_k^2$, respectively, denote the $k^{th}$ variance value and RMS constraints for the variance of the $k^{th}$ subsystem, $\left[\hat{X}_i\right]_{kk}$ denotes the $k^{th}$ diagonal element of the upper bound covariance matrix $\hat{X}_i$, $\left[\hat{X}_i\right]_{kk}$ denotes

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the $k^{th}$ diagonal element of the state covariance $\tilde{X}_j$ which can be defined as

$$\tilde{X}_j = \lim_{t \to \infty} E[ x_i(t) x_j(t) ]$$  \hspace{1cm} (10)

### 3. Controller Design

In this section, we shall first based on Lyapunov-Krasovskii stability theory to develop an algorithm for solving the problem of constructing the RDHC by using LMI approach for the stochastic large-scale uncertain systems with time-delays. It can be also considered as designing a robustly stochastic stabilization subject to $H_\infty$ norm performance constraint. Then, based on the resulting RDHC, a sufficient condition of RDHC subject to state variance constraints can be derived in terms of LMIs. Before proceeding further, we give the following useful lemma for the proof of this work.

**Lemma 3.1** [18]: The linear matrix inequality

$$\begin{bmatrix} Z(\xi) & S(\xi) \\ S(\xi)^T & Y(\xi) \end{bmatrix} > 0$$  \hspace{1cm} (11)

where $Z(\xi)$, $Z^T(\xi)$, $Y(\xi)$, $Y^T(\xi)$, and $S(\xi)$ depend affinely on $\xi$, is equivalent to a nonlinear inequality by Schur complement as

$$Z(\xi) - S(\xi)Y^{-1}(\xi)S(\xi)^T > 0$$  \hspace{1cm} (12)

**Lemma 3.2** [21]: Let $U$, $V$, $W$ and $S(t)$ be real matrices of appropriate dimensions, with $S(t)$ satisfying the norm-bounded condition $S(t)S^T(t) \leq I$, $\forall t$. Then for any matrix $Q > 0$ and scalar $\alpha > 0$, such that the following results both (13) and (14) are hold.

$$U^TQ + Q^TU \leq \alpha Q^TU^TU + \alpha^{-1}V^TV$$  \hspace{1cm} (13)

$$U^TV + V^TU \leq \alpha U^TU + \alpha^{-1}V^TV$$  \hspace{1cm} (14)

### 3.1 The Design of RDHC

We now define the following Lyapunov-Krasovskii functional candidate [7] for the overall interconnected closed-loop system (6):

$$V(x) = \sum_{i=1}^{N} x_i^T(t)P_ix_i(t) + \sum_{j=1}^{N} \int_{-\tau_j}^{0} x_j^T(\theta)\tilde{P}_{ij}x_j(\theta) d\theta$$  \hspace{1cm} (15)

where $P_i$ and $\tilde{P}_{ij}$, $i,j = 1, 2, \ldots, N, j \neq i$, are some positive definite matrices such that $V(x) > 0$. Taking expectation of the time derivative of $V(x)$ along the state trajectory of the system (6) is

$$E \left[ \frac{d}{dt} V(x) \right]$$

$$= \sum_{i=1}^{N} E \left[ x_i^T(t)(\dot{A}_i^T P_i + P_i \dot{A}_i) + (\dot{A}_i^T P_i + P_i \dot{A}_i)x_i(t) \right]$$

$$+ \sum_{j=1}^{N} x_j^T(t)P_j(A_j + \Delta A_j)x_j(t - \tau_j)$$

$$+ x_j^T(t - \tau_j)(A_j + \Delta A_j)^TP_jx_j(t)$$

$$+ \sum_{j=1}^{N} \left( x_j^T(t)\tilde{P}_{ji}x_j(t) - x_j^T(t - \tau_j)\tilde{P}_{ji}x_j(t - \tau_j) \right)$$

$$+ x_j^T(t - \tau_j)\tilde{P}_{ji}x_j(t - \tau_j) + w_j^T(t)D_j^TP_jx_j(t)$$

By Lemma 3.1 and assumption (3), we have

$$x_i^T(t)(\dot{A}_i^T P_i + P_i \dot{A}_i)x_i(t) \leq x_i^T(t)\alpha_{ii}P_iH_i^T P_i + \alpha_{ic}E_i^T E_i$$

$$+ \alpha_{ic}P_iH_i^T P_i + \alpha_{ic}(E_i G_i)^T(E_i G_i)x_i(t)$$

$$x_j^T(t)P_j(A_j + \Delta A_j)^TP_jx_j(t) \leq \alpha_{ij}x_j^T(t)P_jH_j^T P_jx_j(t) + \alpha_{ij}x_j^T(t - \tau_j)E_j^T E_jx_j(t - \tau_j)$$

where $\alpha_{ii} > 0$, $\alpha_{ic} > 0$, and $\alpha_{ij} > 0$. In addition, for some positive matrices $R_{ij}$, $i,j = 1, 2, \ldots, N, j \neq i$, it is always true that

$$x_i^T(t)\tilde{P}_{ij}A_j x_j(t - \tau_j) + x_j^T(t - \tau_j)\tilde{P}_{ij}A_j^Tx_i(t)$$

$$\leq x_i^T(t)\tilde{P}_{ij}A_j R_{ij} A_j^T P_jx_j(t) + x_j^T(t - \tau_j)R_{ij} x_i(t - \tau_j)$$

Let $\tilde{P}_{ij} = \alpha_{ij}E_i^T E_j + R_{ij}$. Then, follows from (16) to (19), a sufficient condition of robustly stochastically stabilizable in probability can be directly obtained with Lyapunov theory as the following quadratic inequality

$$\sum_{i=1}^{N} \tilde{V}_i \leq \sum_{i=1}^{N} E[ x_i^T(t)J_i x_i(t) ] < 0$$  \hspace{1cm} (20)

which implies the following inequality hold,
Obviously, the resulting sufficient condition of robustly stochastic stabilization (20) is not capable of rejecting white noise disturbance. On the other hand, based on the result in (20), we will apply the $H_\infty$ technique, which is still one of the most popular ways to eliminate the external disturbance in the recently literatures, to solve the problem of designing RDHC as presented in the following proposition.

**Proposition 3.1:** Consider the stochastic large-scale uncertain time-delay system (1) satisfying the assumption (3). Then the system is robustly stochastically stabilizable in probability with $H_\infty$ performance level $\gamma_1 > 0$, via state feedback controller (5), for all admissible uncertainties and any time-delays $\tau_{ij} \geq 0$, if there exist some matrices $X_{ii}$, $\Pi_i$, and some positive real number $\alpha_i$, $\beta_i$, and $\gamma_i$, $i, j = 1, 2, \ldots, N, j \neq i$, such that the following LMI condition (22) is satisfied,

$$
\begin{bmatrix}
\Theta_i & X_iE_i^T & L_i & E_i X_i & \bar{X}_i & X_iF_i & D_i \\
E_i X_i & -\alpha_i I & 0 & 0 & 0 & 0 & 0 \\
E_i L_i & 0 & -\alpha_i I & 0 & 0 & 0 & 0 \\
\bar{E}_i X_i & 0 & 0 & -\Gamma_i & 0 & 0 & 0 \\
\bar{X}_i & 0 & 0 & 0 & -\Pi_i & 0 & 0 \\
F_i X_i & 0 & 0 & 0 & 0 & -I & 0 \\
D_i & 0 & 0 & 0 & 0 & 0 & -\gamma_i I
\end{bmatrix} < 0
$$

(22)

Moreover, a RDHC is given by $u_i(t) = G_ix_i(t)$ with $G_i = L_iX_i^{-1}$.

**Proof:** The $H_\infty$ performance constraint (7) can be rewritten as follows:

$$
\sum_{i=1}^{N} \mathbb{E} \left[ \int_0^\infty (z_i^T(t)z_i(t) - \gamma_i^2 w_i^T(t)w_i(t))dt \right] < 0
$$

(24)

Define

$$
\Gamma(t) = \sum_{i=1}^{N} \mathbb{E} \left[ \int_0^\infty (z_i^T(t)z_i(t) - \gamma_i^2 w_i^T(t)w_i(t))dt \right] + \int_0^\infty \tilde{V}_i dt - \mathbb{E} \left[ V(x(\kappa)) - V(x(0)) \right]
$$

(25)

Subject to the zero initial condition $x_i(0) = 0$, we have

$$
\mathbb{E} \left[ V(x(\kappa)) - V(x(0)) \right] = \sum_{i=1}^{N} \int_0^\infty \tilde{V}_i dt
$$

(26)

such that inequality (27) can be hold by $V(x) > 0$,

$$
\Gamma(t) \leq \sum_{i=1}^{N} \mathbb{E} \left[ \int_0^\infty (z_i^T(t)z_i(t) - \gamma_i^2 w_i^T(t)w_i(t))dt \right] + \int_0^\infty \tilde{V} dt
$$

(27)

Substituting the expression of $\sum_{i=1}^{N} \tilde{V}_i$, as defined in (16) into (27) and combining the condition (24), then the following inequality (28) can be obtained by letting $\kappa \to \infty$,

$$
\sum_{i=1}^{N} \mathbb{E} \left[ \int_0^\infty \left( x_i^T(t) \begin{bmatrix} J_i & F_i & P_i & D_i \end{bmatrix} \begin{bmatrix} x_i(t) \\ D_i^T P_i \\ -\gamma_i^2 I \\ w_i(t) \end{bmatrix} dt \right) < 0
$$

(28)

By Schur complement, the quadratic inequality condition (28) is equivalent to,

$$
\begin{bmatrix}
(A_i + B_iG_i)^T P_i + P_i (A_i - B_iG_i) + \alpha_{ii} P_i H_{ii}^T P_i & E_i & L_i & E_i & \bar{X}_i & X_i F_i & D_i \\
E_i & -\alpha_i I & 0 & 0 & 0 & 0 & 0 \\
E_i L_i & 0 & -\alpha_i I & 0 & 0 & 0 & 0 \\
\bar{E}_i X_i & 0 & 0 & -\Gamma_i & 0 & 0 & 0 \\
\bar{X}_i & 0 & 0 & 0 & -\Pi_i & 0 & 0 \\
F_i X_i & 0 & 0 & 0 & 0 & -I & 0 \\
D_i & 0 & 0 & 0 & 0 & 0 & -\gamma_i I
\end{bmatrix} < 0
$$

(23)

Moreover, for $\sum_{i=1}^{N} (P_i A_i R_i A_i^T P_i + R_i^{-1} + \alpha_{ii} P_i H_{ii}^T P_i + \alpha_{ii} E_i^T E_i) < 0$,

$$
F_i F_i + \gamma_i^2 P_i D_i D_i^T P_i < 0
$$

(29)
Pre- and post-multiplying $X_i = P_i^{-1}$ for the both sides of (29) and using $L_i = G_i X_i$, then, the proof is completed by applying Schur complement to (29) again.

3.2 The Design of RDHC with State Variance Constraint

Based on the resulting RDHC from (22), if we further impose the state variance constraint, the design requirements are to be formulated as a convex optimization problem as well. Therefore, the problem of RDHC subjected to upper bound state variance constraint can be directly solved by the following proposition.

**Proposition 3.2:** Consider the desired upper bound on the individual state variance of the system (6) as described in (9). Given $\sigma_i^2 > 0, k = 1, 2, ..., n_i, i = 1, 2, ..., N$, if there exists the positive definite matrix $X_i$ such that the following LMI condition (30) is satisfied,

$$
\begin{bmatrix}
-(\sigma_i^2) & l_i X_i \\
X_i l_i^T & -X_i
\end{bmatrix} < 0, \quad i = 1, ..., N
$$

(30)

where the row vector $l_i \in R^{1 \times n_i}$ with the $k^{th}$ element equal to 1 and others equal to zero. Then, the upper bound of state variance constraint can be achieved.

**Proof:** Rewriting (9), one has

$$l_i^n X_i l_i^T \leq (\sigma_i^2), \quad k = 1, 2, ..., n_i, \quad i = 1, 2, ..., N
$$

(31)

or equivalently,

$$-(\sigma_i^2) + l_i^n X_i (X_i^{-1}) X_i l_i^T \leq 0
$$

(32)

Using the property of Schur complement, the inequality (32) can be reformulated in the LMI form as shown in (30). Therefore, the proof is complete.

Now, the above derivation for RDHC with state variance constraints performance can be summarized into the following main theorem.

**Main theorem:** Consider system (1) that satisfies the assumption (3), given $\gamma_i > 0$ and $\sigma_i^2 > 0, k = 1, 2, ..., n_i, i = 1, 2, ..., N$. Then, the system (1) is robustly stochastically stabilizable in probability with objectives (i) and (ii) held via state-feedback controller (5) for all admissible uncertainties and any time-delays, if there exist some matrices $X_i = X_i^{-1} > 0, R_i = R_i^T > 0, L_i$ and some positive real number $\alpha_{ii}, \alpha_{ih}$ and $\alpha_{ij}, i, j = 1, 2, ..., N, j \neq i$, satisfying the LMI conditions (22) and (30).

**Proof:** From the proofs of Proposition 3.1 and 3.2, one knows that the objective performance (i) and (ii) can be achieved by the suitable convex optimization problem as shown in LMIs (22) and (30). In other words, if matrices $X_i$ and $L_i$ exist and satisfy these LMIs, then the control feedback gain achieve the objective performance constraints (i) and (ii) and can be synthesized by $G_i = L_i X_i^{-1}$. The proof is complete.

4. Numerical Example

A numerical example to demonstrate the effectiveness of the proposed RDHC with state variance constraint control approach for the stochastic large-scale uncertain time-delay systems is given in this section. Consider the stochastic uncertain time-delay systems consisting of three subsystems as follows:

The 1st subsystem:

$$
\begin{align*}
\dot{x}_1(t) &= (A_1 + \Delta A_1)x_1(t) + (B_1 + \Delta B_1)u_1(t) \\
&\quad + (A_{12} + \Delta A_{12})x_2(t - \tau_{12}) \\
&\quad + (A_{13} + \Delta A_{13})x_3(t - \tau_{13}) + D_1 w_1(t), \\
z_1(t) &= F_1 x_1(t)
\end{align*}
$$

(33)

The 2nd subsystem:

$$
\begin{align*}
\dot{x}_2(t) &= (A_2 + \Delta A_2)x_2(t) + (B_2 + \Delta B_2)u_2(t) \\
&\quad + (A_{21} + \Delta A_{21})x_1(t - \tau_{21}) \\
&\quad + (A_{23} + \Delta A_{23})x_3(t - \tau_{23}) + D_2 w_2(t), \\
z_2(t) &= F_2 x_2(t)
\end{align*}
$$

(34)

The 3rd subsystem:

$$
\begin{align*}
\dot{x}_3(t) &= (A_3 + \Delta A_3)x_3(t) + (B_3 + \Delta B_3)u_3(t) \\
&\quad + (A_{31} + \Delta A_{31})x_1(t - \tau_{31}) \\
&\quad + (A_{32} + \Delta A_{32})x_2(t - \tau_{32}) + D_3 w_3(t), \\
z_3(t) &= F_3 x_3(t)
\end{align*}
$$

(35)

where the states $x_i(t) = [x_{i1}(t) \ x_{i2}(t)]^T$, $x_2(t) = [x_{21}(t) \ x_{22}(t)]^T$, $x_3(t) = [x_{31}(t) x_{32}(t)]^T$ and the system matrices $A_i = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 0 \\ 0.2 & 1 \end{bmatrix}$, $A_{13} = \begin{bmatrix} 0 & 0 \\ 0.3 & 1 \end{bmatrix}$, $D_i = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$.
in which $r(t)$, $i(t)$ and $\mu_i(t)$ satisfy (2), and the time-delay parameters given as $\tau_{ij} = 5$, $i, j = 1, 2, 3, \ldots$, $N, j \neq i$, respectively. Then, the proposed objectives design can be carried out as follows:

**Step 1:** From assumption (3), the various known matrices are:

$$H_{i1} = H_{i2} = H_{i3} = H_{i4} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, E_{ia} = E_{2a} = E_{3a} = H_{i1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E_{12} = E_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, E_{21} = E_{23} = E_{31} = E_{32} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$E_{15} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{25} = E_{35} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, S_{ia}(t) = \text{diag} \{r(t)\}, S_{ia}(t) = \text{diag} \{i(t)\}, S_{ia}(t) = \text{diag} \{\mu_i(t)\}, i, j = 1, 2, 3, j \neq i.$$

**Step 2:** The state feedback gain matrices $G_1$, $G_2$ and $G_3$ that achieve the above design specifications (36) and (37) for the closed-loop systems of (33), (34) and (35) can be constructed by using GEVP method in the MATLAB LMI control toolbox subject to the LMI conditions (22) and (30) as,

$$G_1 = [-40.9501, -19.0847]$$

$$G_2 = [-19.9314, 39.4560]$$

$$G_3 = [-19.6218, 25.6259]$$

with optimal performance values

$$(\gamma_1)_{\text{min}} = 0.3102, (\gamma_2)_{\text{min}} = 0.9063, (\gamma_3)_{\text{min}} = 0.5838$$

**Step 3:** The complete control laws for each subsystem are then,

$$u_1(t) = [-40.9501, -19.0847]x_1(t)$$

$$u_2(t) = [-19.9314, 39.4560]x_2(t)$$

$$u_3(t) = [-19.6218, 25.6259]x_3(t)$$

By substituting the control laws (40), (41) and (42) into the corresponding subsystem (33), (34) and (35) then, the frequency responses of each subsystem are shown in Figure 1 to Figure 3, respectively. In which the dotted lines denote the designed upper bounds and solid lines denote the actual value of $H_{\infty}$ norm subject to frequency changed. From Figure 1 to Figure 3, one knows that the $H_{\infty}$ norm performance specifications (39) are well satisfied. The time responses of each subsystem are shown in Figure 4 to Figure 6, where the dotted lines are the zero mean, unitary variance noise input sequences $w_1(t)$, $w_2(t)$ and $w_3(t)$ generated by the MATLAB command, and solid lines stand for the states $x_1(t)$, $x_2(t)$ and $x_3(t)$ response. In addition, the resulting variances of states $\text{Var}(x_{11}) = 0.1656, \text{Var}(x_{12}) = 0.6039, \text{Var}(x_{21}) = 0.1856, \text{Var}(x_{22}) = 0.6349, \text{Var}(x_{31}) = 0.1768, \text{Var}(x_{32}) = 0.6123$. 

The desired control objectives according to (7) and (9) are specified as follows:

**Objective (i):** The $H_{\infty}$ norm constraints are

$$\|H_1(s)\|_\infty \leq (\gamma_1)_{\text{min}}, \quad \|H_2(s)\|_\infty \leq (\gamma_2)_{\text{min}}, \quad \|H_3(s)\|_\infty \leq (\gamma_3)_{\text{min}}$$

(36)

where $(\gamma_1)_{\text{min}}$, $(\gamma_2)_{\text{min}}$ and $(\gamma_3)_{\text{min}}$ are parameters to be minimized during design.

**Objective (ii):** The state covariance upper bound constraints are

$$\text{Var}(x_{ii}) \leq \sigma_i^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2, \quad \text{Var}(x_{jj}) \leq \sigma_j^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2$$

(37)

$$\text{Var}(x_{ii}) \leq \sigma_i^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2, \quad \text{Var}(x_{jj}) \leq \sigma_j^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2, \quad \text{Var}(x_{ij}) \leq \sigma_j^2$$

(38)

(39)

Suppose that $x_1(0) = [4, 7]^T, x_2(0) = [5, 8]^T, x_3(0) = [3, 6]^T$, the white noises $w_1(t), w_2(t), w_3(t)$ satisfying (2), and the time-delay parameters given as $\tau_{ij} = 5$, $i, j = 1, 2, 3, \ldots$, $N, j \neq i$. Then, the proposed objectives design can be carried out as follows:

**Step 1:** From assumption (3), the various known matrices are:

$$H_{1a} = H_{2a} = H_{3a} = H_{1b} = H_{2b} = H_{3b} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, H_{21} = H_{23} = H_{32} = H_{35} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_{13} = H_{15} = H_{21} = H_{23} = H_{32} = H_{35} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_{12} = H_{31} = H_{23} = H_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
0.3475, \( \text{Var}(x_{22}) = 0.6500 \), \( \text{Var}(x_{31}) = 0.2902 \), and \( \text{Var}(x_{32}) = 0.6465 \), respectively, satisfy the individual variance constraint given in (37).

5. Conclusion

This paper has studied the problem of RDHC with state variance constraint performance for the stochastic large-scale uncertain time-delay systems. It has been shown that the RDHC with state variance constraint performance for interconnected stochastic uncertain time-delay systems can be designed via a set of linear matrix inequalities is solvable. Ultimately, a numerical example has shown the effectiveness of the proposed approach. In the further study, we will explore the results of this paper to some practical high performance complex systems.

6. Notations

- \( \Lambda > B \): \( \Lambda - B \) is positive definite
- \( \text{diag}\{\cdot\} \): diagonal matrix of \{\cdot\}
- \( \text{E}[\cdot] \): expectation operator
- \( I \): identity matrix with appropriate dimensions
- \( L_2[0, \infty) \): space of square-integrable vector functions over \([0, \infty)\)
$\mathbb{R}^n$ \hspace{1cm} $n$-dimensional Euclidean space

$\mathbb{R}^{nm}$ \hspace{1cm} the set of $n \times m$ dimensional real matrix

$\cdot^T$ \hspace{1cm} transpose of the vector or matrix $\cdot$

$\| \cdot \|_2$ \hspace{1cm} Euclidean vector norm

$L_2[0, \infty)$ \hspace{1cm} norm

$\lambda_{\min}(A)$ \hspace{1cm} minimum eigenvalue of matrix $A$

$(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ \hspace{1cm} a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$

**References**


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