Ion Slip Effect on Unsteady Couette Flow with Heat Transfer under Exponential Decaying Pressure Gradient

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Abstract

The unsteady Couette flow of an electrically conducting, viscous, incompressible fluid bounded by two parallel non-conducting porous plates is studied with heat transfer taking the ion slip into consideration. An external uniform magnetic field and a uniform suction and injection are applied perpendicular to the plates while the fluid motion is subjected to an exponential decaying with time pressure gradient. The two plates are kept at different but constant temperatures while the Joule and viscous dissipations are included in the energy equation. The effect of the ion slip and the uniform suction and injection on both the velocity and temperature distributions is examined.

Key Words: Unsteady Flow, Couette Flow, Conducting Fluid, Heat Transfer

1. Introduction

The magnetohydrodynamic flow between two parallel plates, known as Hartmann flow, is a classical problem that has many applications in magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays. Hartmann and Lazarus [1] studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary, and insulated plates. Then, a lot of research work concerning the Hartmann flow has been obtained under different physical effects [2–10]. In most cases the Hall and ion slip terms were ignored in applying Ohm’s law as they have no marked effect for small and moderate values of the magnetic field. However, the current trend for the application of magnetohydrodynamics is towards a strong magnetic field, so that the influence of electromagnetic force is noticeable [5]. Under these conditions, the Hall current and ion slip are important and they have a marked effect on the magnitude and direction of the current density and consequently on the magnetic force term. Tani [7] studied the Hall effect on the steady motion of electrically conducting and viscous fluids in channels. Soudalgekar et al. [8, 9] studied the effect of the Hall currents on the steady MHD Couette flow with heat transfer. The temperatures of the two plates were assumed either to be constant [8] or to vary linearly along the plates in the direction of the flow [9]. Abo-El-Dahab [10] studied the effect of Hall current on the steady Hartmann flow subjected to a uniform suction and injection at the bounding plates. Later, Attia [11] extended the problem to the unsteady state with heat transfer, taking the Hall effect into consideration while neglecting the ion slip.

In the present study, the transient flow and heat transfer of an incompressible, viscous, electrically conducting fluid between two infinite non-conducting horizontal porous plates are studied with the consideration of both the Hall current and ion slip. The upper plate is moving with a constant velocity while the lower plate is kept stationary. The fluid is acted upon by an exponential decaying with time pressure gradient, a uniform suction and
injection and a uniform magnetic field perpendicular to the plates. The induced magnetic field is neglected by assuming a very small magnetic Reynolds number \[4,5\]. The two plates are maintained at two different but constant temperatures. The governing equations of are solved numerically taking the Joule and the viscous dissipations into consideration in the energy equation. The effect of the magnetic field, the Hall current, the ion slip, and the suction and injection on both the velocity and temperature distributions is studied.

2. Description of the Problem

The two non-conducting plates are located at the \(y = \pm h\) planes and extend from \(x = -\infty\) to \(+\infty\) and \(z = -\infty\) to \(+\infty\) as shown in Figure 1. The lower and upper plates are kept at the two constant temperatures \(T_1\) and \(T_2\), respectively, where \(T_2 > T_1\). The upper plate is moving with a constant velocity \(U_o\) while the lower plate is kept stationary. The fluid flows between the two plates under the influence of an exponential decaying with time pressure gradient \(dP/dx\) in the \(x\)-direction which is a generalization of a constant pressure gradient. A uniform suction from above and injection from below with constant velocity \(v_o\) which all are applied at \(t = 0\). The whole system is subjected to a uniform magnetic field \(B_o\) in the positive \(y\)-direction. This is the total magnetic field acting on the fluid since the induced magnetic field is neglected. From the geometry of the problem, it is evident that all quantities do not depend on \(x\) or \(z\). The existence of the Hall term gives rise to a \(z\)-component of the velocity. Thus, the velocity vector of the fluid is

\[
\mathbf{v}(y,t) = u(y,t)\mathbf{i} + v(y,t)\mathbf{j} + w(y,t)\mathbf{k}
\]

with the initial and boundary conditions \(u = w = 0\) at \(t \leq 0\), and \(u = w = 0\) at \(y = -h\) and \(u = U_o\) and \(w = 0\) at \(y = h\) for \(t > 0\). The temperature \(T(y,t)\) at any point in the fluid satisfies both the initial and boundary conditions \(T = T_1\) at \(t \leq 0\), \(T = T_2\) at \(y = +h\), and \(T = T_1\) at \(y = -h\) for \(t > 0\). The fluid flow is governed by the momentum equation

\[
\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla P + \mathbf{J} \times \mathbf{B}_o
\]

where \(\rho\) and \(\mu\) are, respectively, the density and the coefficient of viscosity of the fluid. If the Hall and ion slip terms are retained, the current density \(\mathbf{J}\) is given by

\[
\mathbf{J} = \sigma \left[ \nabla \times \mathbf{B}_o - \beta \mathbf{J} \times \mathbf{B}_o + \frac{Bi}{B_o} \left( \nabla \times \mathbf{B}_o \right) \times \mathbf{B}_o \right]
\]

where \(\sigma\) is the electric conductivity of the fluid, \(\beta\) is the Hall factor and \(Bi\) is the ion slip parameter \[4\]. This equation may be solved in \(\mathbf{J}\) to yield

\[
\frac{\mathbf{J} \times \mathbf{B}_o}{(1 + BiB_o)^2 + Be^2} \left[ (1 + BiB_o)u + Be \frac{\partial u}{\partial y} \right] + (1 + BiB_o)w - Be u \mathbf{k}
\]

where \(Be = \sigma \beta B_o\) is the Hall parameter \[4\]. Thus, in terms of Eq. (3), the two components of Eq. (1) read

\[
\rho \left( \frac{\partial u}{\partial t} + \rho v_o \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{(1 + BiB_o)^2 + Be^2} \left[ (1 + BiB_o)u + Be w \right]
\]

\[
\rho \left( \frac{\partial w}{\partial t} + \rho v_o \frac{\partial w}{\partial y} \right) = \mu \frac{\partial^2 w}{\partial y^2} - \frac{\sigma B_o^2}{(1 + BiB_o)^2 + Be^2} \left[ (1 + BiB_o)w - Be u \right]
\]

To find the temperature distribution inside the fluid we use the energy equation \[12\]

\[
\frac{\partial T}{\partial t} + \rho v_o \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left[ \lambda \frac{\partial T}{\partial y} \right]
\]
\[
\frac{\partial T}{\partial t} + \rho c v_0 \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]
\]
\[
+ \frac{\sigma B_0^2}{(1 + B_1 B_2)^2 + B_2^2} (u^2 + w^2)
\]
\[(6)\]

where \(c\) and \(k\) are, respectively, the specific heat capacity and the thermal conductivity of the fluid. The second and third terms on the right-hand side represent the viscous and Joule dissipations, respectively.

The problem is simplified by writing the equations in the non-dimensional form. We define the following non-dimensional quantities

\[
\text{Re} = \frac{\rho h U_0}{\mu}, \quad S = \frac{v_o}{U_0}, \quad \text{Pr} = \frac{\mu c}{k}, \quad \text{Ec} = \frac{U_0}{h}, \quad \text{Ha} = \frac{B_0^2 h^2}{\mu}
\]

Re = \( \rho h U_0 / \mu \) is the Reynolds number,

\( S = v_o / U_0 \) is the suction/injection parameter,

\( \text{Pr} = \mu c / k \) is the Prandtl number,

\( \text{Ec} = U_0 / h \) is the Eckert number,

\( \text{Ha} = B_0^2 h^2 / \mu \) where \( \text{Ha} \) is the Hartmann number.

In terms of the above non-dimensional variables and parameters, the basic Eqs. (4)–(6) are written as (the “hats” will be dropped for convenience)

\[
\frac{\partial u}{\partial t} + S \frac{\partial u}{\partial y} = \frac{dP}{dx} - \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial y^2} - \frac{\text{Ha}^2}{\text{Re}(1 + B_1 B_2)^2 + B_2^2} (u - \text{Be} u + \text{Be} w)
\]

\[(7)\]

\[
\frac{\partial w}{\partial t} + S \frac{\partial w}{\partial y} = \frac{1}{\text{Re}} \frac{\partial^2 w}{\partial y^2} - \frac{\text{Ha}^2}{\text{Re}(1 + B_1 B_2)^2 + B_2^2} (w - \text{Be} w - \text{Be} u)
\]

\[(8)\]

\[
\frac{\partial T}{\partial t} + S \frac{\partial T}{\partial y} = \frac{1}{\text{Re} \text{Pr}} \frac{\partial^2 T}{\partial y^2} + \text{Ec} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \frac{\text{Ec} \text{Ha}^2}{\text{Re}(1 + B_1 B_2)^2 + B_2^2} (u^2 + w^2)
\]

\[(9)\]

The initial and boundary conditions for the velocity become

\[
\begin{align*}
u &= w = 0, t \leq 0, u = w = 0, y = 1, u = 1, w = 0, y = 0, t > 0
\end{align*}
\]

\[(10)\]

and the initial and boundary conditions for the temperature are given by

\[
t \leq 0 : T = 0, t > 0 : T = 1, y = 1, T = 0, y = 1
\]

\[(11)\]

where the pressure gradient is assumed in the form \( dP / dx = C e^{-\alpha t} \).

### 3. Numerical Solution of the Governing Equations

Equations (7)–(9) are solved numerically using finite differences [13] under the initial and boundary conditions (10) and (11) to determine the velocity and temperature distributions for different values of the parameters \( \text{Ha}, \text{Be}, \text{Bi} \) and \( S \). The Crank-Nicolson implicit method is applied. The finite difference equations are written at the mid-point of the computational cell and the derivative terms are replaced by their second-order central difference approximations in the \( y \)-direction. The diffusion term is replaced by the average of the central differences at two successive time levels. The viscous and Joule dissipation terms are evaluated using the velocity components and their derivatives in the \( y \)-direction which are obtained from the numerical solution of the momentum equations. Finally, the block tri-diagonal system is solved using Thomas’ algorithm. All calculations have been carried out for \( C = -5, \alpha = 1, \text{Re} = 1, \text{Pr} = 1 \) and \( \text{Ec} = 0.2 \).

### 4. Results and Discussion

Figure 2 shows the profiles of the velocity components \( u \) and \( w \) and temperature \( T \) for various values of time \( t \). The figure is plotted for \( \text{Ha} = 3, \text{Be} = 3, \text{Bi} = 3, \) and \( S = 1 \). As shown in Figure 2a and 2b, the profiles of \( u \) and \( w \) are asymmetric about the plane \( y = 0 \) because of the suction. It is observed that the velocity component \( u \) reaches the steady state faster than \( w \) which, in turn, reaches the steady state faster than \( T \). This is expected, since \( u \) is the source of \( w \), while both \( u \) and \( w \) act as sources for the temperature.

Figure 3 shows the time evolution of \( u \) and \( w \) at the centre of the channel \( y = 0 \) for various values of the Hall parameter \( \text{Be} \) and the ion slip parameter \( \text{Bi} \). In this figure, \( \text{Ha} = 3 \) and \( S = 0 \). It is clear from Figure 3a that increas-
ing the parameter $Be$ or $Bi$ increases $u$. This is because the effective conductivity $\sigma/(1 + Bi Be)^2 + Be^2)$ decreases with increasing $Be$ or $Bi$ which reduces the magnetic damping force on $u$. In Figure 3b, the velocity component $w$ increases with increasing $Be$, since $w$ is a result of the Hall effect. On the other hand, increasing the ion slip parameter $Bi$ decreases $w$ for all values of $Be$ as a result of decreasing the source term of $w$ $(Be Ha^2 u/(1 + Bi Be)^2 + Be^2)$ and increasing its damping term $(Ha^2 w/(1 + Bi Be)^2 + Be^2)$. The influence of the ion slip on $w$ becomes more pronounced for higher values of $Be$. Figure 3c indicates that increasing $Be$ or $Bi$ decreases $T$ for all times. This can be attributed to the fact that, an increase in $Be$ or $Bi$ decreases the Joule dissipation which is also proportional to $(1/(1 + Bi Be)^2 + Be^2))$.

Figure 4 shows the time evolution of $u$ and $w$ at the centre of the channel $y = 0$ for various values of the Hartmann number $Ha$ and the ion slip parameter $Bi$. In this figure, $Be = 3$ and $S = 0$. Figure 4a indicates that the effect of $Bi$ on $u$ depends on $Ha$. For small values of $Ha$, increasing $Bi$ slightly decreases $u$ as a result of increasing the damping force on $u$ which is proportional to $Bi$. Increasing $Bi$ more increases the effective conductivity and, in turn, decreases the damping force on $u$ which increases $u$. On the other hand, for larger values of $Ha$, $u$ becomes small, and increasing $Bi$ always decreases the effective conductivity and therefore increases $u$. It is also
clear that the effect of \( Bi \) on \( u \) becomes more apparent for higher values of \( Ha \). Figure 4b ensures that increasing the ion slip parameter \( Bi \) decreases \( w \) for all values of \( Ha \) and that its effect is more apparent for higher values of \( Ha \). Figures 4c indicates that the parameter \( Bi \) has a more pronounced effect on \( T \) for higher values of the magnetic field. It is clear that, for all \( Ha \), increasing \( Bi \) decreases \( T \) as a result of decreasing the viscous and Joule dissipations.

Figure 5 presents the time evolution of \( u \) and \( w \) at the centre of the channel \( y = 0 \) for various values of the suction parameter \( S \) and the ion slip parameter \( Bi \). In this figure \( Ha = 3 \) and \( Be = 3 \). Figures 5a and 5b show that increasing the suction decreases both \( u \) and \( w \) due to the convection of the fluid from regions in the lower half to the centre which has higher fluid speed. It is also clear from Figures 5a and 5b that the effect of the suction parameter on \( u \) becomes more pronounced as \( Bi \) increases while its effect on \( w \) decreases as \( Bi \) increases. Figure 5c shows that increasing \( S \) decreases the temperature at the centre of the channel. This is due to the influence of convection in pumping the fluid from the cold lower half towards the centre of the channel.
5. Conclusion

The unsteady Couette flow of a conducting fluid under the influence of an applied uniform magnetic field has been studied, considering the Hall and ion slip effects in the presence of uniform suction and injection. The effect of the magnetic field, the Hall parameter, the ion slip parameter, and the suction and injection velocity on the velocity and temperature distributions has been investigated. It is found that the effect of the ion slip on the main velocity \( u \) depends upon the magnetic field. For large values of the magnetic field, increasing the ion slip increases \( u \). For small values of the magnetic field, increasing the ion slip slightly decreases \( u \), but increasing it more increases \( u \). It is also shown that increasing the Hall parameter increases the velocity component \( w \), while increasing the ion slip decreases \( w \). The influence of the Hall current on \( w \) decreases greatly as the ion slip increases. The effect of the suction and injection velocity on \( u \) increases as the ion slip increases while its effect on \( w \) decreases when increasing the ion slip. The influence of the ion slip on the temperature \( T \) depends on the magnetic field and becomes more pronounced for higher values of magnetic field.

References


Manuscript Received: Dec. 6, 2005
Accepted: May. 10, 2006