Differential Quadrature Method for Solving Hyperbolic Heat Conduction Problems

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Abstract

This work analyzes hyperbolic heat conduction problems using the differential quadrature method. Numerical results are compared with published results to assess the efficiency and systematic procedure of this novel approach to solving hyperbolic heat conduction problems. The computed solutions to the hyperbolic heat conduction problems correlate well with published data. Two examples are analyzed using the proposed method. The effects of the relaxation time on the temperature distribution are investigated. Hyperbolic heat conduction problems are solved efficiently using the differential quadrature method.

Key Words: Hyperbolic Heat Conduction, Differential Quadrature Method, Numerical Analysis

1. Introduction


This work attempts to solve hyperbolic heat conduction problems using the proposed differential quadrature method. To our knowledge, this work is the first to solve hyperbolic heat conduction problems using this method. The Chebyshev-Gauss-Lobatto point distribution is utilized. The integrity and computational accuracy of the differential quadrature method in solving this problem are demonstrated through various case studies.

2. Differential Quadrature Method

Differential quadrature was originally developed by simple analogy with integral quadrature, which is derived using the interpolation function [32–34]. Currently, many engineers apply numerical techniques to solve sets of linear algebraic equations. They include the Rayleigh-Ritz method, the Galerkin method, the finite element method, the boundary element method and others, which are applied to solve differential equations. Bellman et al. [32,33] first introduced the differential quadrature method. His work has been applied in diverse areas of computational mechanics and many researchers have claimed that the differential quadrature approach is a highly accurate scheme that requires minimal computational efforts. Differential quadrature has been shown to be a powerful tool for solving initial and boundary value problems, and has thus has become an alternative to existing methods. Bert et al. [34–46], who investigated the static and free vibration of beams and rectangular plates using the differential quadrature method, proposed the δ technique that can be applied to the double boundary conditions of plate and beam problems. In this approach, boundary points are chosen at a small distance. However, δ must be small to ensure the accuracy of the solution, but solutions oscillate when δ is excessively small. Bert et al. [34–46] incorporated boundary conditions using weighted coefficients. In the formulation of the differential quadrature approach, multiple boundary conditions are directly applied to the weighted coefficients and unlike in the δ-interval method, no point need be selected. The accuracy of calculated results will be independent of the δ-interval. Liew et al. [47–50], who analyzed rectangular plates at rest on Winkler foundations using the differential quadrature method, also presented a static analysis of laminated composite plates subjected to transverse loads using the differential quadrature method. The efficiency and accuracy of the Rayleigh-Ritz method depend on the number and accuracy of selected comparison functions. However, the differential quadrature method does not generate such difficulties when appropriate comparison functions are used. The differential quadrature method is a continuous function that can be approximated by a high-order polynomial in the overall domain, and a derivative of a function at a sample point can be approximated as a weighted linear summation of functional values at all sample points in the overall domain of that variable. Using this approximation, the differential equation is then transformed into a set of algebraic equations. The number of equations depends on the number of sample points selected. As for any polynomial approach, increasing the number of sample points increases the accuracy of the solution. Numerical interpolation schemes can be used to eliminate potential oscillations in numerical results that are associated with high-order polynomials. For a function, \( f(x, t) \), the differential quadrature method approximation for the \( m^{th} \) order derivative at the \( i^{th} \) sample point is given as follows [47–50].

\[
\frac{\partial^m}{\partial x^m} \begin{bmatrix} f(x_1, t) \\ f(x_2, t) \\ \vdots \\ f(x_N, t) \end{bmatrix} \equiv \begin{bmatrix} D^m_{y_1} \\ D^m_{y_2} \\ \vdots \\ D^m_{y_N} \end{bmatrix} \begin{bmatrix} f(x_1, t) \\ f(x_2, t) \\ \vdots \\ f(x_N, t) \end{bmatrix} \text{ for } i, j = 1, 2, \ldots, N
\]

(1)

where \( f(x_i, t) \) is the functional value at grid point \( x_i \), and
The weighting coefficient of the $m^{th}$ order differentiation of these functional values. The most convenient technique is to choose equally spaced grid points, but the numerical results were inaccurate in this work when equally spaced points were selected. This finding demonstrates that the choice of a grid point distribution and the test functions markedly influence the efficiency and accuracy of the results in some cases. The selection of grid points always importantly affects solution accuracy. Unequally spaced sample points on each domain have the following Chebyshev-Gauss-Lobatto distribution in the present computation [47–50].

$$x_i = \frac{1}{2} \left( 1 - \cos \left( \frac{(i - 1) \pi}{N - 1} \right) \right) \quad \text{for} \quad i = 1, 2, ..., N$$  \hspace{1cm} (2)

The differential quadrature weighted coefficients can be derived using numerous techniques. To overcome the numerically poor conditions in determining the weighted coefficients, $D_{ij}^{(m)}$, the following Lagrangian interpolation polynomial is introduced [47–50].

$$f(x_i,t) \equiv \sum_{i=1}^{N} \frac{M(x)}{(x-x_i)M_i(x_i)} f(x_i,t)$$  \hspace{1cm} (3)

where

$$M(x) = \prod_{j=1}^{N} (x-x_j)$$

$$M_i(x_j) = \prod_{j=1,j\neq i}^{N} (x_i-x_j) \quad \text{for} \quad i = 1, 2, ..., N$$

Substituting Eq. (3) into Eq. (1) yields the following equations [47–50].

$$D_{ij}^{(1)} = \frac{M_i(x_j)}{M_j(x_i)(x_i-x_j)M_i(x_j)} \quad \text{for} \quad i, j = 1, 2, ..., N \quad \text{and} \quad i \neq j$$  \hspace{1cm} (4)

and

$$D_{ij}^{(0)} = -\sum_{j=1,j\neq i}^{N} D_{ij}^{(0)} \quad \text{for} \quad i = 1, 2, ..., N$$  \hspace{1cm} (5)

Once the grid points are selected, the coefficients of the weighted matrix can be acquired using Eqs. (4) and (5). Notably, the numbers of test functions exceed the highest order of the derivative in the governing equations. High-order derivatives of weighted coefficients can also be acquired using matrix multiplication [47–50], as follows.

$$D_{ij}^{(2)} = \sum_{k=1}^{N} D_{ik}^{(0)} D_{kj}^{(0)} \quad \text{for} \quad i, j = 1, 2, ..., N$$  \hspace{1cm} (6)

$$D_{ij}^{(3)} = \sum_{k=1}^{N} D_{ik}^{(0)} D_{kj}^{(0)} \quad \text{for} \quad i, j = 1, 2, ..., N$$  \hspace{1cm} (7)

$$D_{ij}^{(4)} = \sum_{k=1}^{N} D_{ik}^{(0)} D_{kj}^{(0)} \quad \text{for} \quad i, j = 1, 2, ..., N$$  \hspace{1cm} (8)

3. Analysis

Consider a finite slab and a dimensionless formulation; the hyperbolic equation for a one-dimensional problem has the form [1]

$$\frac{\partial^2 T(x,t)}{\partial t^2} + 2 \frac{\partial T(x,t)}{\partial t} - \frac{\partial^2 T(x,t)}{\partial x^2} = 0 \quad \text{at} \quad t > 0, \quad 0 < x < l$$  \hspace{1cm} (9)

with the initial condition,

$$T(x,0) = 0$$  \hspace{1cm} (10)

$$\frac{\partial T(x,t)}{\partial t} = 0$$  \hspace{1cm} (11)

and the boundary conditions,

$$T(0,t) = 1$$  \hspace{1cm} (12)

$$\frac{\partial T(l,t)}{\partial x} = 0$$  \hspace{1cm} (13)

where $T$ represents the temperature field $T(x,t)$. The differential quadrature method is applied and Eq. (1) is substituted into Eq. (9). The equation discretizes the sample points as,

$$\left[ \begin{array}{c} D_{1}^{(2)} \ D_{2}^{(2)} \ \cdots \ \ D_{N}^{(2)} \end{array} \right] \{T(x_i,t)\} = \{0\}$$  \hspace{1cm} (14)

for $i = 1, 2, ..., N$ and $j = 1, 2, ..., N$.
Equation (12) can be rewritten as
\[
[I \ 0 \ 0 \ \cdots \ 0] \{T(x_j, t)\} = \{1\}
\]
for \( j = 1, 2, \cdots, N \) \hfill (15)

According to the differential quadrature method, Eq. (13) takes the following discrete forms;
\[
\left[ \frac{D^{(1)}_{l}}{l} \ \frac{D^{(2)}_{l}}{l} \ \cdots \ \frac{D^{(N)}_{l}}{l} \right] \{T(x_j, t)\} = \{0\}
\]
for \( j = 1, 2, \cdots, N \) \hfill (16)

Many inner and boundary points in the computational domain contribute directly to the calculation of the derivatives and the state variables, due to the global nature of the differential quadrature method. Many inner and boundary points in the computational domain contribute directly to the calculation of the derivatives and the state variables due to the global nature of the differential quadrature method. Figures 1–3 plot the temperature distributions at \( t = 0.5, 1.0 \) and 1.5, respectively. The numerical results calculated using the differential quadrature method are very consistent with the results in the literature [1] for \( t = 0.5, 1.0 \) and 1.5. Consider a slab with thickness \( l \) and constant thermal properties. This slab originally has a uniform temperature distribution [1]. The adiabatic condition is applied to \( x = 0 \). At a specific time \( t = 0 \), a heat flux \( q(t) \) is applied to \( x = l \). A mathematical formation of the problem is as follows [1].
\[
\beta \rho C \frac{\partial^2 T(x,t)}{\partial t^2} + \rho C \frac{\partial T(x,t)}{\partial t} - k \frac{\partial^2 T(x,t)}{\partial x^2} = 0
\]
at \( t > 0, \ 0 < x < l \) \hfill (17)

The initial conduction are,
\[
T(x,0) = T_0, \ 0 \leq x \leq l
\]
and the boundary conditions are,
\[
\frac{\partial T(x,t)}{\partial x} = 0 \quad \text{at} \quad x = 0, \ t > 0
\]
\[
\beta \frac{\partial q(x,t)}{\partial t} + k \frac{\partial T(x,t)}{\partial x} + q(x,t) = 0 \quad \text{at} \quad x = l, \ t > 0
\]
where \( T \) represents the temperature field \( T(x, t) \), \( k \) is the thermal conductivity, \( \rho \) is the density, \( C \) is the specific heat, \( C \) is the heat capacity per unit volume, and \( \beta \) is the relaxation time, which is nonnegative. The differential quadrature method is applied and Eq. (1) is substituted into Eq. (17). The equation discretizes the sample points as
\[
\left[\beta \rho C \left[ \frac{\partial^2 T(x_j, t)}{\partial t^2} \right] + \rho C \left[ \frac{\partial T(x_j, t)}{\partial t} \right] - k \frac{D^{(1)}_{l}}{l^2} \frac{D^{(2)}_{l}}{l^2} \cdots \frac{D^{(N)}_{l}}{l^2} \right] \{T(x_j, t)\} = \{0\}
\]
for \( i = 1, 2, \cdots, N \) and \( j = 1, 2, \cdots, N \) \hfill (22)
Equation (20) can be rewritten as

\[
\left[ \frac{D(1)}{l} \frac{D(1)}{l} \frac{D(1)}{l} \ldots \frac{D(1)}{l} }{0} \right] \{ T(x_j, t) \} = \{ 0 \}
\]

for \( j = 1, 2, \ldots, N \) (23)

In the differential quadrature method, Eq. (21) takes the following discrete forms:

\[
[\beta] \left[ \frac{\partial T(x_j, t)}{\partial t} \right] + \left[ \frac{D(1)}{l} \frac{D(1)}{l} \frac{D(1)}{l} \ldots \frac{D(1)}{l} }{0} \right] \{ T(x_j, t) \} + \{ q(x_j, t) \} = \{ 0 \} \quad \text{for} \quad j = 1, 2, \ldots, N
\]

(24)

Many inner and boundary points in the computational domain are used directly in the calculation of the derivatives and the state variables due to the global nature of the differential quadrature method. Consider a slab with thickness \( l = 0.035 \) m and constant thermal properties \( k = 50 \, \text{W/m}^2 \) and \( k/\rho C = 0.0001327 \, \text{m}^2/\text{sec} \) [1]. Figure 4 plots the input heat flux [1]. A time-varying heat flux is applied at the side \( x = l \) between time steps 200 and 230 and the magnitude of the heat flux varies at each time step. Figures 5–9 depict the direct solution when \( \beta = 0, 1, 10, 100, \) and 1000. The values of \( l, k, \rho \) and \( C \) are set to unity in Eq. (17). The strength of the heat source is presented as a time-varying function. Figures 5–9 plot the numerical results between time steps 200 to 230. The curvature increases and reduces the temperature, depending on the relaxation time \( \beta \). The speed of heat propagation varies with the relaxation time \( \beta \). The magnitude of the thermal properties significantly affects the analysis of the temperature distribution. The numerical results show that the temperature wave is sensitive to the
relaxation time. The first investigation yields good results concerning the use of the differential quadrature method for solving hyperbolic heat conduction problems.

4. Conclusion

This work proposes the differential quadrature method for solving hyperbolic heat conduction problems. The partial differential equation is reduced to a set of algebraic equations using the differential quadrature method. The considered examples demonstrate that the present method is in good agreement with solutions taken from the literature [1]. Simulation results verify that the differential quadrature method yields accurate results with relatively little computational and modeling effort.

The magnitude of the thermal properties significantly influences the temperature distribution. The relaxation time $\beta$ affects the temperature distribution. The demonstrated simplicity and accuracy of this formulation makes it a good candidate for modeling various problems in science and engineering.

References


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