Propagation of Channel Wave in an Incompressible Anisotropic Initially Stressed Plate of Finite Thickness

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Abstract

This paper deals with the propagation of channel waves in an incompressible anisotropic initially stressed plate. Attempt has been made to find out the effect of anisotropy and initial stresses on the velocity of propagation of channel waves. Some particular cases have been discussed. The numerical values of velocities of channel wave for different parameters (anisotropy and initial stresses) have been calculated and the results have been presented in a number of tables.

Key Words: Channel Wave, Initial stress, Anisotropy

1. Introduction

The term ‘initial stress’ is meant by stresses already developed in a medium before it is being use for study. Some times it is called pre-stress. This initial stress may be developed in a medium due to natural phenomena or by any artificial process. Due to variation in temperature, atmospheric pressure, gravity, manufacturing process, slow process of creep, weight dropping, pin pointing etc, enormous amount of initial stresses may exist inside the earth. So, earth is considered to be an initially stressed body. Biot [1] mentioned that initial stresses have remarkable effect on the propagation of elastic waves in a medium. He has shown that the propagation of waves in an initially stressed body is fundamentally different from classical theory of elasticity through a great extent. The problem of finite deformation of an elastic body and the effect of high initial stress on wave propagation were discussed in a series of investigations by Murnaghan [2] and others.

It is well established fact that the earth is an initially stressed medium, stresses of very high magnitude are present in the layers of the earth. These initial stresses contribute a significant influence on propagation of seismic waves produced by earthquakes, explosions or impact. Basically there are two types of seismic waves - (i) Body waves and (ii) Surface waves. Body waves radiated by the source propagated in all directions (free waves) while surface waves start to propagate first after body waves of different types have been interacting along the boundaries of different layers. This surface waves are always concentrated near discontinuity of surface and are therefore some times called boundary waves or guided waves. In other words for homogeneous media i.e. for media with no boundaries, there is no surface wave propagation. The body waves which travel faster than surface waves are of two types - (i) Compress ional (longitudinal) and (ii) Shear waves (transverse waves) [3].

Using the Biot’s theory [4] several investigators [5–11] have studied extensively the propagation of surface waves such as Rayleigh waves, Love waves, Edge waves etc. The influence of initial stresses on the propagation of edge waves has been discussed by Das and Dey [7], Dey and Addy [9]. The authors of this paper also shown that the velocity of propagation not only influenced by anisotropy and initial stress parameters but also it depends highly on the direction of propagation of shear waves.
The waves propagated through the central part of a region are called Channel waves. Enough literature is available [12–14] on different waves as discussed above but, very less is obtained about channel wave propagation. In the present paper authors have shown that both the parameters anisotropy and initial stresses have significant effect on the propagation of channel waves. The study reveals that in an isotropic initial stress free thick plate the velocity of channel wave propagation becomes twice the velocity of shear waves. Also in case of an initially stressed free isotropic thin plate, it has been observed that the velocity of channel wave propagation is zero, that means channel wave propagation does not exist.

2. Formulation of the Problem

Consider an elastic plate of thickness 2h composed of incompressible elastic medium with different rigidities N and Q along x and y directions respectively. Let the initial compressive stresses $S_{11}$ and $S_{22}$ be present in the medium and acting along x and y directions respectively (Figure 1). Let us assume that the origin of the coordinate system be located in the middle of the layer and y axis be taken positive vertically upwards as shown in the Figure 1.

The dynamical equations of motion [4] for the propagation of wave in two dimensions are given by

$$\frac{\partial^2 s_{11}}{\partial x^2} + \frac{\partial^2 s_{22}}{\partial y^2} + p \frac{\partial^2 \omega}{\partial y^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 s_{12}}{\partial x \partial y} + \frac{\partial^2 s_{22}}{\partial y^2} + p \frac{\partial^2 \omega}{\partial x \partial y} = \rho \frac{\partial^2 v}{\partial t^2}$$

where $\rho$ is the density of the medium and $u$, $v$ are the displacement components along x and y directions respectively and

$$P = S_{22} - S_{11}$$

further $\omega$ is the rotational component, i.e.

$$\omega = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

The incremental stress-strain relations for the incompressible anisotropic medium may be written as

$$S_{11} - s = 2N e_{xx}, \quad s_{22} - s = 2N e_{yy}, \quad s_{12} = 2Q e_{xy}$$

where $s_{ij}$ are the incremental stresses i.e. $s_{11}$ and $s_{22}$ are incremental normal stresses along x and y directions respectively and $s_{12}$ is the incremental shear stress in the plane of xy. N and Q are rigidity along x and y directions respectively.

and

$$s = \frac{s_{11} + s_{22}}{2}$$

Also the incremental strains are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

where $e_{xx}$ & $e_{yy}$ are incremental normal strains and $e_{xy}$ is

Figure 1. A plate of finite thickness 2h under initial stresses. (Geometry of the problem)
the incremental shear strain in the plane of \( xy \). Also the condition for incompressibility is

\[
e_{xx} + e_{yy} = 0
\]

which is satisfied by the following relations

\[
u = \frac{\partial \phi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial x}
\]  \( (5) \)

where \( \phi(x,y) \) is a differentiable function.

### 3. Solution of the Problem

The equation of motion (1) with the help of (2), (3) and (4) may be expressed in terms of wave potential \( \phi \) as

\[
\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} \left( 2N - Q + \frac{P}{2} \frac{\partial^2 \phi}{\partial y^2} + \left( Q + \frac{P}{2} \frac{\partial^2 \phi}{\partial y^2} \right) \right) = -\rho \frac{\partial^2 \phi}{\partial t^2}
\]

\[
\frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial x^2} \left( 2N - Q + \frac{P}{2} \frac{\partial^2 \phi}{\partial x^2} + \left( Q - \frac{P}{2} \frac{\partial^2 \phi}{\partial x^2} \right) \right) = \rho \frac{\partial^2 \phi}{\partial t^2}
\]  \( (6) \)

Elimination of \( s \) from the two equations in (6), one gets

\[
\left( \frac{Q + P}{2} \right) \frac{\partial^4 \phi}{\partial y^2 \partial^2 x^2} + \left( 4N - 2Q \right) \frac{\partial^4 \phi}{\partial y^2 \partial x^4} + \left( Q - \frac{P}{2} \frac{\partial^2 \phi}{\partial x^2} \right) \frac{\partial^2 \phi}{\partial x^2} = -\rho \frac{\partial^2 \phi}{\partial t^2}
\]

\[
= \rho \left( \frac{\partial^4 \phi}{\partial x^2 \partial^2 y^2} + \frac{\partial^4 \phi}{\partial y^2 \partial x^2} \right)
\]  \( (7) \)

For harmonic wave propagation along the direction of \( x \)-axis, the solution of (7) may be written as

\[
\phi(x,y) = \frac{1}{F} f(ly) \sin(\alpha x - \alpha t)
\]  \( (8) \)

where \( \frac{\alpha}{t} = c \), velocity of propagation of wave in the medium.

Using equation (8) in equation (6) one gets

\[
s = F(ly) \cos(\alpha x - \alpha t)
\]  \( (9) \)

where

\[
F(ly) = \left( \frac{2N - Q + \frac{P}{2} \frac{\partial^2 \phi}{\partial y^2}}{Q - \frac{P}{2} \frac{\partial^2 \phi}{\partial x^2}} \right) \frac{\partial^4 \phi}{\partial y^2 \partial x^2} + \left( Q + \frac{P}{2} \right) f^* - \left( Q + \frac{P}{2} \right) f^*
\]  \( (10) \)

and dashes denote the order of differentiation with respect to \( ly \).

It is obvious that channel waves occupy throughout the channel without reduction of the amplitude, the value of \( f(ly) \) may be assumed as

\[
f(ly) = c_1 \sin \beta_1 ly + c_2 \sin \beta_2 ly
\]  \( (11) \)

where \( c_1 \) and \( c_2 \) are two constants and \( \beta_1 \) and \( \beta_2 \) are real roots of the following non-dimensional equation

\[
f^{iv} - \left( \frac{4N}{Q} - Q - 2 \frac{\rho \alpha^2}{Q \ell^2} \right) f^* + \left( 1 + \frac{P}{2Q} \right) \left( 1 - \frac{P}{2Q} \right) f = 0
\]

\( (12) \)

which may be written in the following form

\[
f^{iv} - 2m f^* + k^2 f = 0
\]  \( (13) \)

where primes denote the differentiation with respect to \( ly \) and

\[
2m = \left( \frac{4N - 2 \frac{\rho \alpha^2}{Q \ell^2}}{1 + \frac{P}{2Q}} \right)
\]

and

\[
k^2 = \left( 1 - \frac{P}{2Q} \right) - \frac{\rho \alpha^2}{Q \ell^2}
\]

which gives

\[
giving the values of \( \beta_1 \) and \( \beta_2 \) as

\[
\beta_1^2 = -m - \sqrt{m^2 - k^2}
\]

\[
\beta_2^2 = -m + \sqrt{m^2 - k^2}
\]  \( (14) \)

In order that \( \beta_1 \) and \( \beta_2 \) are real

\[
m^2 \geq k^2
\]

which gives
for real values of \( \rho\alpha^2/Q^2 \) one get

\[
\left(1 - \frac{N}{Q}\right) \left(1 + \frac{P}{2Q}\right) \geq 0
\]

(15)

Inequality (15) must be satisfied in order to get the real values of the velocity of propagation.

### 3. Boundary Conditions

The geometry of the boundary forces per unit area in an initially stressed body is given in Figure 2.

The left side figure shows the position of a body under initial stress just before deformation and the right side figure shows the same body under initial stress just after deformation.

Considering the free surfaces at the top and bottom, the boundary conditions may be written as

\[
\Delta f_x = 0 \quad \text{at} \quad y = \pm h;
\]
\[
\Delta f_y = 0 \quad \text{at} \quad y = \pm h
\]

(16)

where \( \Delta f_x \) and \( \Delta f_y \) are the incremental boundary forces as given in the Figure 2 and are given by the following expressions [4],

\[
\Delta f_x = \sigma_{12} - s_{22}\omega - s_{11}\epsilon_{11};
\]
\[
\Delta f_y = s_{22} + s_{22}\epsilon_{xx}
\]

(17)

On substituting the values and after simplification the equations (17) gives

\[
\frac{\Delta f_x}{L} = \frac{Q}{2} \left[ \left( \frac{2 - \sigma_{22} + \sigma_{11}}{Q} \right) f + \left( \frac{2 + \sigma_{22} - \sigma_{11}}{Q} \right) f'' \right] \sin(\omega x - \alpha t)
\]

\[
\frac{\Delta f_y}{L} = \left[ \left( \frac{4N - 1 + \frac{P}{2Q} - \frac{\rho\alpha^2}{Q^2}}{1 + \frac{P}{2Q}} \right) f' + \left( \frac{2 + \sigma_{22} - \sigma_{11}}{Q} \right) f'' \cos(\omega x - \alpha t) \right]
\]

(18)

where

\[
L = 1 + \frac{P}{2Q}
\]

Since \( f \) and its derivatives involve with the terms \( \omega x \) then \( \Delta f_x \) and \( \Delta f_y \) may be represented as

\[
\Delta f_x = \tau_1(\omega y) \sin(\omega x - \alpha t)
\]
\[
\Delta f_y = \tau_2(\omega y) \cos(\omega x - \alpha t)
\]

(19)

Therefore, the boundary conditions (16) may be expressed as

\[
\frac{\tau_1(\omega y)}{L} = 0 \quad \text{at} \quad y = \pm h
\]
\[
\frac{\tau_2(\omega y)}{L} = 0 \quad \text{at} \quad y = \pm h
\]

(20)

Substitution of the values of \( f, f', f'', f''' \) in (18), we get
For non-zero solution of \( c_1 \) and \( c_2 \) one gets the determinant of the coefficient matrix of \( c_1 \) and \( c_2 \) is equal to zero. On expansion, this determinant gives

\[
\frac{\tau_1(llh)}{L} = \left[ \begin{array}{c}
\frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} & \frac{2 - \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \\
\frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} & -\frac{2 + \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}}
\end{array} \right] \beta_1 \sin \beta_1 llh + \left[ \begin{array}{c}
\frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} & -\frac{2 + \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \\
\frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} & \frac{2 + \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}}
\end{array} \right] \beta_2 \sin \beta_2 llh = 0
\]

(21)

Equation (23) with condition given in (15) is the velocity equation of channel wave propagation in an initially stressed anisotropic incompressible medium.

4. Particular Cases

In order to get some information in connection with the propagation of channel waves the following particular cases have been discussed.

**Case I: When the plate is very thin**

\[ \tan \beta_1 llh \rightarrow \beta_1 llh \]
\[ \tan \beta_2 llh \rightarrow \beta_2 llh \]

and the velocity equation (20) becomes

\[
\frac{\tau_1(llh)}{L} = \left[ \begin{array}{c}
\frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} & \frac{2 - \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \\
\frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} & -\frac{2 + \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}}
\end{array} \right] \beta_1 \cos \beta_1 llh + \left[ \begin{array}{c}
\frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} & -\frac{2 + \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \\
\frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} & \frac{2 + \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}}
\end{array} \right] \beta_2 \cos \beta_2 llh = 0
\]

(22)

which on simplification gives

\[
\left( \frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} + \frac{2 - \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \right) \beta_1 + \left( \frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} + \frac{2 - \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}} \right) \beta_2 \tan \beta_2 llh = 0
\]

(23)

Substituting the value of \( (\beta_1^2 - \beta_2^2) \), one gets.

\[
\left( \frac{2 - \frac{s_{22} + s_{11}}{Q}}{1 + \frac{P}{2Q}} + \frac{2 - \frac{s_{22} - s_{11}}{Q}}{1 + \frac{P}{2Q}} \right) \beta_1 \tan \beta_1 llh + \left( \frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} + \frac{2 - \frac{s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}} \right) \beta_2 \tan \beta_2 llh = 0
\]

(24)

The above result (25) shows that if the plate is very thin then the velocity of channel wave propagation depends on both anisotropy and initial stress parameters.

**Case I(a): In addition to the above condition if the plate is free from initial stresses**

i.e. \( S_{11} = S_{22} = P = 0 \), equation (25) yields
which gives three velocities of propagation as given below,

\[
\frac{c}{\beta} = 2 \sqrt{\frac{N}{Q}} \quad \text{or} \quad \left( \frac{4N - \alpha^2}{\beta^2} \right) - 4 \left( 1 - \frac{\alpha^2}{\beta^2} \right) = 0
\]

Thus when the plate is very thin and it is free from initial stress then the velocity of channel propagation depends on the anisotropic factor \(N/Q\).

**Case I(b):** Considering the medium be isotropic and free from initial stresses

\[S_{11} = S_{22} = P = 0\]

and \(N = Q\)

the equation (26) takes the form

\[
\left( 4 - \frac{\alpha^2}{\beta^2} \right) \times \frac{c^4}{\beta^4} = 0
\]

giving \(c = 2\beta\), or \(c = 0\). Now in a very thin isotropic initially stressed free plate the velocity of channel wave propagation can not become the twice the shear wave velocity. Hence the other value is true that is \(c = 0\), which shows that when the plate is very thin and isotropic then in absence of initial stress channel wave does not propagate in the medium.

**Case I(c):** Considering the presence of initial stresses and the medium be isotropic

\[N = Q\]

the equation (25) takes the form

\[
\left( \frac{2 + \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} \right) + \left( \frac{2 + \frac{-s_{22} - s_{11}}{P}}{1 + \frac{P}{2Q}} \right) \left( \frac{3 + \frac{P}{2Q}}{1 + \frac{P}{2Q}} \right) \left( \frac{4N - 1 - \frac{\alpha^2}{\beta^2}}{1 + \frac{P}{2Q}} \right) \left( \frac{1}{1 + \frac{P}{2Q}} \right) = 0
\]

This shows that the velocity propagation depends upon initial stresses.

**Case II: When the plate is very thick**

\[\tan \beta_1 l = 1\]

\[\tan \beta_2 l = 1\]

Then the velocity equation (23) takes the form

\[
\left( \frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} \right) + \left( \frac{4N - 1 + \frac{P}{2Q}}{1 + \frac{P}{2Q}} \right) \left( \frac{1 + \frac{P}{2Q}}{1 + \frac{P}{2Q}} \right) \left( \frac{1}{1 + \frac{P}{2Q}} \right) = 0
\]

Thus when the plate is very thick the velocity of channel wave propagation depends on both the parameters initial stress and anisotropy.

**Case II(a):** In addition to the above condition if the medium is free from initial stresses

\[S_{11} = S_{22} = P = 0\]

the equation (29) gives

\[
\left( \frac{4N - 1 - \frac{\alpha^2}{\beta^2}}{1 + \frac{P}{2Q}} \right) \beta_1^2 + \left( \frac{4N - 1 - \frac{\alpha^2}{\beta^2}}{1 + \frac{P}{2Q}} \right) \beta_2^2 = \left( \frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} \right) \beta_1 + \left( \frac{2 - \frac{s_{22} + s_{11}}{P}}{1 + \frac{P}{2Q}} \right) \beta_2
\]

On simplification the above equation gives

\[
\frac{c^6}{\beta^6} - 8 \frac{N}{Q} \frac{c^4}{\beta^4} + \left( \frac{16N^2}{Q^2} - 8\frac{N}{Q} + 8 \right) \frac{c^2}{\beta^2} + \left( 16 - 64\frac{N}{Q} + 48 \frac{N^2}{Q^2} \right) = 0
\]
In this case the velocity of channel wave propagation depends only on the anisotropic parameter $N/Q$.

Case II(b): In addition to the above condition if the medium be isotropic ($N = Q$), then the equation (30) gives

$$\frac{c^6}{\beta^6} - 8 \frac{c^4}{\beta^4} + 16 \frac{c^2}{\beta^2} = 0 \quad (31)$$

which gives either $c = 0$ or $c = 2\beta$

This shows that in an isotropic initial stress free thick plate either the velocity of propagation becomes zero or it becomes twice the velocity of shear waves in the medium.

Case II(c): In case of isotropic medium ($N = Q$) with existing initial stresses, by substituting $N = Q$ and $S_{22} = P + S_{11}$, equation (31) gives

$$\left\{ \frac{2 - P + 2S_{11}}{Q} \right\} \left\{ \frac{3 + P - \frac{c^2}{\beta^2} - P + S_{11}}{2Q} \right\} \beta_1 + \left\{ \frac{2 - P + 2S_{11}}{Q} \right\} \beta_1^2$$

This shows that in an isotropic thick plate the velocity of propagation depends on initial stress parameter only.

5. Numerical Computations and Discussion

The numerical computation shows that the wave does not propagate in the medium when $N/Q$ is less than one.

Table 1. When the medium is isotropic ($N = Q$) i.e. $R = 1$ and $S_{22}/2Q = 0.0$

<table>
<thead>
<tr>
<th>$H$</th>
<th>0.5</th>
<th>1.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{11}/2Q$</td>
<td>0.00</td>
<td>0.40</td>
<td>0.50</td>
</tr>
<tr>
<td>$c / \beta$</td>
<td>0.10</td>
<td>0.90</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 2. When $H = 2.0$ and $S_{22}/2Q = 0.0$

<table>
<thead>
<tr>
<th>$R$</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{11}/2Q$</td>
<td>0.50</td>
</tr>
<tr>
<td>$c / \beta$</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 3. When the medium is isotropic $R = N/Q = 1$ and $S_{11}/2Q = 0.0$

<table>
<thead>
<tr>
<th>$H$</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{22}/2Q$</td>
<td>-0.50</td>
<td>0.00</td>
<td>-0.50</td>
</tr>
<tr>
<td>$c / \beta$</td>
<td>1.00</td>
<td>0.10</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 4. When $H = 2.0$ and $S_{11}/2Q = 0.0$

<table>
<thead>
<tr>
<th>$R$</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{22}/2Q$</td>
<td>-0.50</td>
</tr>
<tr>
<td>$c / \beta$</td>
<td>1.00</td>
</tr>
</tbody>
</table>

6. Conclusion

The study reveals that the presence of initial stresses and anisotropy plays an important role in the propagation of Channel wave in a medium. The anisotropy factor as well as the initial stress parameter have dominant effect on the velocity of propagation. In the present paper authors have shown that both the parameters anisotropy and initial stresses have significant effect on the propagation of channel waves. The study shows that in absence of initial stress, the velocity of channel wave pro-
pagation through an isotropic thick plate, becomes twice the velocity of shear waves. Also it has been observed that channel wave does not propagate in an isotropic initially stressed free very thin plate.

References


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