A Note on Ranks of the Difference of Orthogonal Projections from Unitary Orbits

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Abstract

The unitary orbit of an \( n \times n \) complex matrix \( A \) is the set consisting of matrices unitarily similar to \( A \). In this note we offer an alternative proof for a recent result, due to Li, Poon and Sze, on the possible ranks of the difference of matrices taken from the unitary orbits of two given orthogonal projections.

Key Words: Rank, Orthogonal Projection, Unitary Matrix, Unitary Orbit

1. Introduction

Denote by \( M_n \) the set of \( n \times n \) complex matrices. Let \( A \in M_n \). The unitary orbit of \( A \) is defined and denoted by \( \mathfrak{U}(A) = \{ U^*AU \mid U^*U = I_n \} \). Clearly, if \( A \) is unitary similar to a diagonal matrix (equivalently, \( A \) is a normal matrix), then such diagonal matrix must belongs to \( \mathfrak{U}(A) \). In the literature there are many results involved with the unitary orbit set, see [1–6] and their references. In this note we focus the discussion on the orthogonal projection of which the unitary orbit contains a diagonal matrix with 0 and 1 as its only possible diagonal entries.

In [1] Li, Poon and Sze studied ranks (as well as determinants) of matrices of the form \( X + Y \) with \( X \in \mathfrak{U}(A) \) and \( Y \in \mathfrak{U}(B) \) where \( A, B \in M_n \) are given. In particular, it was proved that

\[
\max \{ \operatorname{rank}(X + Y) \mid X \in \mathfrak{U}(A), Y \in \mathfrak{U}(B) \} = \min \{ m, n \},
\]

where \( m = \min \{ \operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n) \mid \mu \in \mathbb{C} \} \) (see [1, Theorem 2.1]). It was noted that a metric on the set of unitary orbit of \( A \) is given by

\[
d(\mathfrak{U}(A), \mathfrak{U}(B)) = \min \{ \operatorname{rank}(X - Y) \mid X \in \mathfrak{U}(A), Y \in \mathfrak{U}(B) \};
\]

however, it is not easy to determine the minimum rank

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matrices $U$, $V$ are chosen such that $U^* AU = I_p \oplus 0_{n-p}$ and $V^* BV = I_q \oplus 0_{n-q}$. It is asserted that $\text{rank}(U^* AU - V^* BV)$ is equal to $k$. Clearly, the “if” part of Theorem 1 follows if the assertion holds. Notice that if $k = |p - q| + 2j \leq \min\{p + q, 2n - p - q\}$, then

$$j \leq \min\{p, n - q\} \text{ or } j \leq \min\{q, n - p\},$$

according to $p \leq q$ or $p > q$, respectively; or equivalently, $j \leq \min\{p, q, n - p, n - q\}$.

Therefore, the index $j$ in theorem 1 has to satisfy $0 \leq j \leq \min\{p, q\}$ and the original proof of the “if” part is correct when $j \leq p \leq q$. Since without loss of generality, one can always assume $p \leq q$. Thus the original proof of the “if” part indeed works if the authors add the assumption $p \leq q$ at the beginning.

To establish the “only if” part of Theorem 1 we need the following lemmas.

**Lemma 1**: Let $S$ and $T$ be subspaces of $\mathbb{C}^n$, then

$$\dim S - \dim(S \cap T) - \dim(S \cap T^\perp) = \dim S - \dim(S^\perp \cap T) - \dim(S^\perp \cap T^\perp).$$

**Proof**: Since

$$n - \dim(S^\perp \cap T) = \dim(S^\perp \cap T^\perp) = \dim(S + T^\perp) = \dim(S) + \dim(T^\perp) - \dim(S \cap T^\perp),$$

we have

$$\dim(S^\perp) = n - \dim(S) = \dim(T^\perp) - \dim(S \cap T^\perp),$$

and therefore

$$\dim S^\perp - \dim(S^\perp \cap T) - \dim(S^\perp \cap T^\perp) = \dim(T^\perp) - \dim(S \cap T^\perp) - \dim(S^\perp \cap T^\perp) = \dim(T^\perp) - \dim(S \cap T^\perp) - \dim(S + T^\perp) = \dim S - \dim(S \cap T) - \dim(S \cap T^\perp).$$

For a matrix $A$, denote the range space and the null space of $A$ respectively, by $R(A)$ and $N(A)$.

**Lemma 2**: Let $A$, $B$ be $n \times n$ orthogonal projections. Then $N(A - B) = (R(A) \cap R(B)) \oplus (N(A) \cap N(B))$, and $\text{rank}(A - B) - |\text{rank}A - \text{rank}B|$ is a nonnegative even integer.

**Proof**: Since $A$, $B$ are orthogonal projections, it is clear that $(R(A) \cap R(B)) + (N(A) \cap N(B))$ is a direct sum and is included in $N(A - B)$. Conversely, let $x \in N(A - B)$. Clearly, we have $Ax = Bx$ and $x - Ax = x - Bx$. But $x = Ax + (x - Ax)$, so $x \in (R(A) \cap R(B)) + (N(A) \cap N(B))$. This establishes the equality $N(A - B) = (R(A) \cap R(B)) \oplus (N(A) \cap N(B))$.

Let $\dim(R(A) \cap R(B)) = r$, $\dim(R(A) \cap N(B)) = s$, $\dim(N(A) \cap R(B)) = u$, and $\dim(N(A) \cap N(B)) = v$. By Lemma 1 (with $S, T$ equal to $R(A)$ and $R(B)$ respectively), $\dim(R(A) - r - s$, $\dim(N(A) - u - v$, $\dim(R(B) - r - u$ and $\dim(N(B) - s - v$ are all equal. Let $t$ denote the common value. Then

$$\text{nullity}(A - B) = \dim(R(A) \cap R(B)) + \dim(N(A) \cap N(B)) = r + v,$$

and

$$n = \dim R(A) + \dim N(A) = (r + s + t) + (u + v + t) = r + s + u + v + 2t,$$

where the second equality holds by the definition of $t$, and so

$$\text{rank}(A - B) = n - \text{nullity}(A - B) = s + u + 2t.$$

Hence

$$\text{rank}(A - B) - |\text{rank}A - \text{rank}B| = s + u - |s - u| + 2t$$

is a nonnegative even integer.

Proof of the “only if” part of Theorem 1:

Consider $U^* AU - B$ for a given unitary matrix $U$. Let $k = \text{rank}(U^* AU - B)$. The inequality $k \leq \min\{p + q, 2n - p - q\}$ is obvious because

$$\text{rank}(U^* AU - B) \leq \text{rank}(U^* AU) + \text{rank}(B) = p + q$$

and

$$\text{rank}(U^* AU - B) = \text{rank}(U^* (I - A)U - (I - B)) \leq \text{rank}(I - A) + \text{rank}(I - B) = 2n - p - q.$$
The fact $k = |p - q| + 2j$ for some nonnegative integer $j$, can be readily seen from Lemma 2. □

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References


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