Optimum Process Mean Setting for Product with Rework Process

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Abstract

The determination of the optimum process mean has been a major topic in statistical process control. It directly affects the process defective rate, production cost, scrap cost, and rework cost. In 2000, Lee et al. presented a filling problem for determining the optimum process mean and screening limits. They considered three grades of product, assumed a normal quality characteristic, and adopted the piecewise linear profit function for measuring the profit per item. However, they have not included the scrap cost and the perfect rework process in their model. In this paper, we further propose a modified Lee et al.’s model with rework process for determining the optimum process mean. Both perfect rework and imperfect rework processes for the product are considered in the model.

Key Words: Process Mean, Process Variance, Rework Process

1. Introduction

The optimum process mean setting has been a major topic in modern statistical process control. It may not be equal to the target value because the costs of below and above the specification limits are different. The determination of the optimum process mean should achieve the minimum expected cost per item or the maximum expected profit per item.

There is considerable attention paid to the study of economic selection of the process mean. Recently, Li \([1–5]\), Li and Chirng \([6]\), Li and Cherng \([7]\), Li and Chou \([8]\), Li and Wu \([9,10]\), Wu and Tang \([11]\), Maghsoudloo and Li \([12]\), and Phillips and Cho \([13]\), have addressed different problems of unbalanced tolerance design with the asymmetric quadratic and linear quality loss functions.

The piecewise linear profit function of the quality characteristic is usually applied in the filling/canning problem for determining the optimum manufacturing target and other important parameters, see for example, Springer \([14]\), Hunter and Kartha \([15]\), Carlsson \([16,17]\), Bisgaard et al. \([18]\), Golhar \([19,20]\), Golhar and Pollock \([21,22]\), Rahim and Banerjee \([23]\), Arcelus and Rahim \([24]\), Boucher and Jafari \([25]\), Al-Sultan \([26]\), Pulak and Al-Sultan \([27]\), Al-Sultan and Al-Fawzan \([28]\), Al-Sultan and Pulak \([29]\), Lee and Jang \([30]\), Misiorek and Barnett \([31]\), Lee and Elsayed \([32]\), Lee et al. \([33,34]\), and Duffuaa and Siddiqi \([35]\).

In Misiorek and Barnett’s \([31]\) model, the aim is to fix the filling mean of the process in order to maximize the expected profit per container. The profit for a container depends on the filling value of the material, i.e., whether or not it is over-filled, under-filled, or rejected. Misiorek and Barnett’s \([31]\) model considered that the expense of recapturing over-filled material is a cost per unit, the expense of emptying out under-filled containers and putting the material back into the process is a constant cost, and that the containers from under-filled items are discarded. They also considered the expected profit per container for the following four special cases: (1) the

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containers from under-filled items are discarded, (2) the containers from under-filled items are re-used, (3) the containers from under-filled items are discarded and there is no overflow, and (4) all under-filled containers are topped-up and over-flowed material is captured.

Lee et al. [33,34] presented the problem of a joint determination of optimum process mean and screening limits. The quality characteristic of the performance variable or surrogate variable is considered as the screening variable. Their models involved selling and discounted prices as well as production, inspection, rework and penalty costs. The normal and bivariate normal distributions are assumed and used in Lee et al.’s [33,34] models. The screening of a product with three grades, using single stage screening and two stage screening are considered. The objective of their models is to maximize the expected profit per item.

For the filling/canning industry, the product needs to be produced within the specification limits. The manufacturing cost per unit considers the fixed and variable production costs and the constant inspection cost. The variable production cost is proportional to the value of the quality characteristic. A product usually cannot be sold at a higher price for the constant label content. If a product is above the upper specification limit (USL), it will cause an increment in the manufacturing cost. This is not a desirable situation for the production department. If a product is below the lower specification limit (LSL), it will cause a loss of goodwill for the manufacturer. The company may face customers’ claims or penalty due to government’s laws. This is not a desirable case for the marketing department. Thus, the canning manufacturing industry needs product conformance. The penalty cost due to loss of goodwill is usually higher than the finite manufacturing cost. Hence, a product is usually put to scrap when it is below LSL and put to rework when it is above the USL.

Taguchi [36, pp. 27–32] proposed the optimum tolerance design with 100% inspection under the assumptions of no scrap and perfect rework for product. However, it is hard to get an overall perfect product in the production process. Hence, in our paper, we consider the possibility of the filled product having rework, either perfect or imperfect, and scrap.

Lee et al. [33] proposed the inspection of three grades of product and adopted the piecewise linear profit function for measuring the profit per item. However, they have not included the scrap cost and perfect reprocessing in their model. In this paper, we propose a modified Lee et al.’s [33] model with rework process for determining the optimum process mean. The production cost, inspection cost, rework cost and scrap cost are included in the modified model. Both perfect rework and imperfect rework process for the product are considered. A numerical example and sensitivity analysis of parameters are provided for illustration. Previous researchers addressed a product scrap that is sold at a reduced price in the market. Our modified model addressed the case that a scrapped product cannot be sold in the market, instead it involves a scrap cost. These are the main differences between our model and the Lee et al.’s [33] model.

2. Literature Review — Lee et al.’s Model

In Lee et al.’s [33] model, which considers performance as a variable, the objective is to maximize the expected profit per item and obtain the optimum process mean. The profit for an item depends on the value of a normal quality characteristic, Y. Each item is classified into three grades A, B, and C. Grade A items are sold to primary market and grade B items are sold to secondary market. Grade C items are reworked and the rework process is the same as the original production process. Let \( L_1 \) be the pre-specified specification limit for grade A and \( L_2 \) be the pre-specified specification limit for grade B, where \( L_2 < L_1 \). If an item has \( y > L_1 \), it is sold at a fixed price \( a_1 \) to the primary market. If an item has \( L_2 \leq y \leq L_1 \), it is sold at a fixed price \( a_2 ( < a_1 ) \) to the secondary market. If an item has \( y < L_2 \), it is reworked by the same production process at a rework cost \( r ( < a_2 < a_1 ) \).

It is assumed that the quality characteristic \( Y \) is normally distributed with an unknown process mean \( \mu \) and a known standard deviation \( \sigma \). Let the production cost per item be \( b + cy \), where \( b \) is the fixed production cost and \( c \) is the variable production cost per item. Let \( i \) be the inspection cost per item and \( y_r \) be the quality characteristic of a reworked item. It is assumed that \( y \) and \( y_r \) are identically and independently distributed.
From Lee et al. [33, p. 181], we have the profit per item as follows:

\[ P(y) = \begin{cases} \ a_i - b - cy - i, & y > L_1 \\ \ a_i - b - cy - i, & L_2 \leq y \leq L_1 \\ \ P(y) - r - i, & y < L_2 \end{cases} \]  

(1)

Assume that the reworked item has the same profit as the non-reworked item, i.e., \( P(y) = P(y_r) \). Hence, the expected profit per item is

\[ E[P(y)] = \int_{L_1}^{\infty} (a_i - b - cy - i) f(y) dy \]

\[ + \int_{L_2}^{L_1} (a_i - b - cy - i) f(y) dy \]

\[ + \int_{-\infty}^{L_2} (E[P(y)] - r - i) f(y) dy \]  

(2)

where \( f(y) \) is the normal probability density function of \( Y \).

From Lee et al. [33, p. 181], the above Eq. (2) can be rewritten as

\[ E[P(y)] = \left\{ \begin{array}{ll} a_i \Phi\left( \frac{\mu - L_1}{\sigma} \right) + a_i \Phi\left( \frac{L_1 - \mu}{\sigma} \right) - \Phi\left( \frac{L_1 - \mu}{\sigma} \right) \\ - (b + c_i) \Phi\left( \frac{\mu - L_2}{\sigma} \right) - r \Phi\left( \frac{L_2 - \mu}{\sigma} \right) \\ - c_i \Phi\left( \frac{\mu - L_2}{\sigma} \right) - i / \Phi\left( \frac{\mu - L_2}{\sigma} \right) \end{array} \right\} \]  

(3)

Lee et al. [33] took the first derivative of Eq. (3) with respect to \( \mu \), set it equal to zero, and adopted the bisection method for finding the optimal \( \mu \) that maximizes the expected profit per item.

3. Modified Lee et al.’s Model with Scrap Cost

All items are inspected prior to shipment to the customers. If an item has \( Y > M_1 \), it is reworked by the same production process at a rework cost \( r \). Items with \( Y < M_2 \) are scrapped at a scrap cost \( s \). Items with \( M_2 \leq Y \leq M_1 \) are shipped to the market at a price \( a \). The production cost per item is linear in \( Y \), that is \( b + cy \), where \( b \) is the fixed production cost and \( c \) is the variable production cost. Let \( i \) be the inspection cost per item.

3.1 Perfect Rework

Assume that the rework process is perfect. The perfect rework model, which include production, inspection, rework and scrap costs, is as follows:

\[ TP_i = \begin{cases} \ a - b - cy - i - r, & y > M_1 \\ \ a - b - cy - i, & M_2 \leq y \leq M_1 \\ \ 0 - b - cy - i - s, & y < M_2 \end{cases} \]  

(4)

From Eq. (4), the expected profit per item is

\[ E(TP_i) = \int_{M_1}^{\infty} (a - b - cy - i - r) f(y) dy \]

\[ + \int_{M_2}^{M_1} (a - b - cy - i) f(y) dy \]

\[ + \int_{-\infty}^{M_2} (0 - b - cy - i - s) f(y) dy \]  

\[ = a - b - c_i - i - r [1 - \Phi\left( \frac{M_1 - \mu}{\sigma} \right)] - (a + s) \Phi\left( \frac{M_2 - \mu}{\sigma} \right) \]  

(5)

To find the optimum process mean \( \mu^* \), Eq. (5) is differentiated with respect to \( \mu \) and set equal to 0, giving:

\[ E(TP_i) = -c + r \frac{M_1 - \mu}{\sigma} + \frac{a + s}{\sigma} \phi\left( \frac{M_2 - \mu}{\sigma} \right) \]

If the second derivative of Eq. (5) is negative, that is,

\[ a + s \frac{M_2 - \mu}{\sigma} - r (M_1 - \mu) \frac{1}{\sigma^2} \phi\left( \frac{M_1 - \mu}{\sigma} \right) < 0, \]

then \( \mu^* \) is optimal. We can use Mathematica for obtaining the optimum process mean.

3.2 Imperfect Rework

Consider an imperfect process. The reworked product may be scrapped, perfect, or reworked again. The rework process can be continued. It is assumed that the quality characteristic of any reworked product is the same as that of the original process. The imperfect rework model, which include production cost, inspection cost, rework cost, and scrap cost is as follows:

\[ TP_i = \begin{cases} \ E(TP_i) - b - cy - i - r, & y > M_1 \\ \ a - b - cy - i, & M_2 \leq y \leq M_1 \\ \ 0 - b - cy - i - s, & y < M_2 \end{cases} \]  

(6)
From Eq. (6), the expected total cost per item is

\[
E(TP_2) = \int_{M_1}^{M_2} [E(TP_2) - b - cy_i - r] f(y) dy \\
+ \int_{M_1}^{M_2} (a - b - cy_i) f(y) dy + \int_{-\infty}^{M_2} (0 - b - cy_i - s) f(y) dy \\
= E(TP_2)[1 - \Phi(M_2 - \mu)] - b - c\mu - i \\
- r[1 - \Phi(M_1 - \mu)] + a\Phi(M_2 - \mu) - (a + s)\Phi(M_1 - \mu) \\
- \Phi(M_1 - \mu)] \\
(7)
\]

Eq. (7) can be rewritten as

\[
E(TP_2) = r + a - b + c\mu + i + r - \frac{(a + s)\Phi(M_2 - \mu)}{\Phi(M_1 - \mu) - \Phi(M_1 - \mu)} \\
(8)
\]

To find the optimum process mean \( \mu^* \), Eq. (8) is differentiated with respect to \( \mu \) and set equal to 0, giving:

\[
E'(TP_2) = -\frac{[c - \frac{a + s}{\sigma}(\Phi(M_2 - \mu))] - \frac{[b + c\mu + i + r + (a + s)\Phi(M_2 - \mu)](\Phi(M_1 - \mu) - (a + s)\Phi(M_1 - \mu))}{\sigma(\Phi(M_1 - \mu))^2}}{\sigma(\Phi(M_1 - \mu))^2} \\
(9)
\]

If the second derivative of Eq. (9) is negative, that is,

\[
2\frac{c - \frac{a + s}{\sigma}(\Phi(M_2 - \mu))\Phi(M_1 - \mu)}{\sigma(\Phi(M_1 - \mu))^2} \\
+ \frac{[a + s](M_2 - \mu)\Phi(M_2 - \mu)]}{\sigma(\Phi(M_1 - \mu))^2} \\
[\frac{b + c\mu + i + r + (a + s)\Phi(M_2 - \mu)](\Phi(M_1 - \mu) - (a + s)\Phi(M_1 - \mu))}{\sigma(\Phi(M_1 - \mu))^2} \\
0
\]

then \( \mu^* \) is optimal. We can use Mathematica for obtaining the optimum process mean.

From Appendix A, we have the following conclusion about the perfect and imperfect rework models: if the \( \mu \) value of both models are identical, then the expected profit per item of the former is always larger than or equal to that of the latter.

4. Numerical Example and Sensitivity Analysis

4.1 Numerical Example

Consider the packing plant of a tea drink. The plant consists of two processes: an inspection process and a filling process. Inspection is performed by measuring the ingredients of the tea drink. Assume that the ingredients of the tea drink is above the USL which increases the manufacturing cost and that the tea drink cannot be sold at a higher price. Hence, the producer adopts a rework for it. If the ingredients of the tea drink is below the LSL, a penalty cost due to government’s law may occur. Hence, the producer adopts a scrap for it. For the rework of a product, there exists the perfect and imperfect rework cases. Conforming ingredients of the tea drink is canned by a filling machine and moved to the dispatching stages on a conveyor belt. From theoretical considerations and past experience, it is known that the ingredients of the tea drink \( Y \) is normally distributed with a known standard deviation \( \sigma = 0.25 \) and an unknown mean \( \mu \). Let the target value of the mean be 40.75. Assume that the cost components and the specification limits for \( Y \) are \( a = 5, s = 0.3, r = 0.1, b = 0.1, c = 0.06, i = 0.04, M_1 = 41.5 \) and \( M_2 = 40 \). The producer would like to determine the optimum process mean for maximizing the expected profit per item.

By solving Eq. (5), the optimum process mean for the perfect rework model is \( \mu^* = 40.783 \) with \( E(TP_1) = 2.4082 \). By solving Eq. (8), the optimum process mean for the imperfect rework model is \( \mu^* = 40.742 \) with \( E(TP_2) = 2.4042 \). In this example, the perfect rework model has a larger process mean and expected profit than those of the imperfect rework model. The computational results agree with our intuition. The perfect rework assumes that the error rate falls to zero during rework. The perfect rework model only reworks the product once.
4.2 Sensitivity Analysis

Figures 1–7 show the change in the values of the parameters and their effects on the process mean and the expected profit per item for both the perfect and imperfect rework models. From Figures 1–7, we have the following conclusions:

1. The optimum process mean of the perfect rework model is larger than that of the imperfect rework model. The

Note: 1. \( \mu_1 \) and \( \mu_2 \) denote the process mean for the perfect and imperfect rework, respectively.
2. \( E(TP_1) \) and \( E(TP_2) \) denote the expected profit per item for the perfect and imperfect rework model, respectively.

The little difference between \( E(TP_1) \) and \( E(TP_2) \) is not visible in the figure.

**Figure 1.** The effect of \( a \) for the perfect and imperfect model.

Note: \( \mu_1 \) and \( \mu_2 \) denote the process mean for the perfect and imperfect rework, respectively; \( E(TP_1) \) and \( E(TP_2) \) denote the expected profit per item for the perfect and imperfect rework model, respectively.

**Figure 2.** The effect of \( s \) for the perfect and imperfect model.

Note: \( \mu_1 \) and \( \mu_2 \) denote the process mean for the perfect and imperfect rework, respectively; \( E(TP_1) \) and \( E(TP_2) \) denote the expected profit per item for the perfect and imperfect rework model, respectively.

**Figure 3.** The effect of \( \sigma \) for the perfect and imperfect model.
value of the optimum process mean increases as the selling price \( a \), scrap cost \( s \), or process standard deviation \( \sigma \) increases. It decreases as the rework cost \( r \) or variable production cost \( c \) increases. Its value is stable as the fixed production cost \( b \) or inspection cost \( i \) increases.

2. The expected profit of the perfect rework model is larger than or equal to that of the imperfect rework model. The difference is significant when the value of

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**Figure 4.** The effect of \( r \) for the perfect and imperfect model.

**Figure 5.** The effect of \( b \) for the perfect and imperfect model.

**Figure 6.** The effect of \( c \) for the perfect and imperfect model.
scrap cost $s$, process standard deviation $\sigma$, or rework cost $r$ changes. The difference is not significant when the value of the selling price $a$, variable production cost $c$, fixed production cost $b$ or inspection cost $i$ changes.

3. The value of the expected profit of both models increases as the selling price $a$ increases. Otherwise, it decreases. The value of the expected profit of both models changes fast when the selling price $a$ or variable production cost $c$ changes.

Hence, they have a major effect on the expected profit.

5. Conclusion

Lee et al. [33] considered a filling problem when items are produced continuously and that they have three grades, where every item is inspected. In Lee et al.’s [33] model, there is no scrap for the product. However, in real world situations, a product may be scrapped when an internal failure cost occurs. In this paper, we have proposed a modified Lee et al.’s [33] model with scrap cost for determining the optimum process mean. The extension to the modified Lee et al.’s [33] model with a skewed quality characteristic or asymmetric quality loss function may be left for further study.

Nomenclature

- $a$ : the selling price in the modified Lee et al.’s model
- $a_1$ : the selling price for the primary market in the Lee et al.’s model
- $a_2$ : the selling price for the secondary market in the Lee et al.’s model
- $b$ : the fixed production cost per item
- $c$ : the variable production cost per item
- $E(TP_1)$ : the expected profit per item for the perfect rework model
- $E(TP_2)$ : the expected profit per item for the imperfect rework model
- $f(y)$ : the normal probability density function, $f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$, $-\infty < y < \infty$
- $i$ : the inspection cost per item
- $L_1$ : the pre-specified specification limit for the item with grade A in the Lee et al.’s model
- $M_1$ : the upper specification limit in the modified Lee et al.’s model
- $L_2$ : the pre-specified specification limit for the item with grade B in the Lee et al.’s model
- $M_2$ : the lower specification limit in the modified Lee et al.’s model
- $P(y)$ : the profit per item for the Lee et al.’s model
- $P(y_r)$ : the profit for a rework item for the Lee et al.’s model
- $r$ : the rework cost for the item with grade C in the Lee et al.’s model and the rework cost in the modified Lee et al.’s model
- $s$ : the scrap cost in the modified Lee et al.’s model
- $TP_1$ : the profit per item for the perfect rework
- $TP_2$ : the profit per item for the imperfect rework model

Note: $\mu_1$ and $\mu_2$ denote the process mean for the perfect and imperfect rework, respectively; $E(TP_1)$ and $E(TP_2)$ denote the expected profit per item for the perfect and imperfect rework model, respectively.

Figure 7. The effect of $i$ for the perfect and imperfect model.
the quality characteristic of the performance variable

$y_r$ the quality characteristic of a reworked item, where it is assumed that $y$ and $y_r$ are independent and identically distributed

$\mu$ the unknown process mean

$\sigma$ the known process standard deviation

$\Phi(z)$ the cumulative probability of a standard normal random variable with a probability density function:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

### Appendix A. A Comparison of the Expected Profit of Perfect and Imperfect Rework Models

From Eqs. (5) and (8), we have

$$E(TP_1) = a - b - c\mu - i - r[1 - \Phi(M_1 - \mu, \sigma)] - (a + s)\Phi(M_2 - \mu, \sigma)$$

(A1)

$$E(TP_2) = r + a - \frac{b + c\mu + i + r}{\Phi(M_1 - \mu, \sigma)} - \frac{(a + s)\Phi(M_2 - \mu, \sigma)}{\Phi(M_1 - \mu, \sigma)}$$

(A2)

Assume that both Eqs. (A1) and (A2) have the same $\mu$ value. Hence,

$$E(TP_1) - E(TP_2) = a - b - c\mu - i - r[1 - \Phi(M_1 - \mu, \sigma)] - (a + s)\Phi(M_2 - \mu, \sigma)$$

$$- [r + a - \frac{b + c\mu + i + r}{\Phi(M_1 - \mu, \sigma)} - \frac{(a + s)\Phi(M_2 - \mu, \sigma)}{\Phi(M_1 - \mu, \sigma)}]$$

$$= \frac{(a + s)\Phi(M_2 - \mu, \sigma)}{\Phi(M_1 - \mu, \sigma)} - \frac{(a + s)\Phi(M_2 - \mu, \sigma)}{\Phi(M_1 - \mu, \sigma)} \geq 0$$

(A3)

Note that $0 \leq \Phi(\cdot) \leq 1$. If the $\mu$ value of both models are identical, then the expected profit per item of the perfect rework model is always larger than or equal to that of the imperfect rework model.

### References


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