Frequency Synchronization plus Channel Estimation by Least Squares Approach for OFDM in Frequency-Selective Channels

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Abstract

Applying the least squares technique, we perform frequency synchronization and channel estimation by using repeated identical blocks of training data for orthogonal frequency division multiplexing (OFDM) systems in frequency-selective channels. Analyses on the performances of our estimations are given. Then, the symbol error rate (SER) performance for QAM transmission in Rayleigh fading channels under imperfect estimations is investigated and compared with a closed-form expression for the ideal SER with perfect estimations. Simulation results are found in good agreements with theoretical predictions and prove that our algorithm provides good estimators.

Key Words: Orthogonal Frequency Division Multiplexing (OFDM), Carrier Frequency Offset (CFO), Least-Squares Estimation, Doppler Spread, Channel Estimation, Synchronization

1. Introduction

Various techniques of frequency synchronization in OFDM systems have been proposed in the literature [1–11]. Most of these techniques apply the maximum-likelihood (ML) estimation algorithm. When applying the ML techniques, to perform frequency synchronization requires channel knowledge. Likewise, to perform channel estimation, the frequency offset must be given. Further, the highly nonlinear nature of the log-likelihood function often results in very complicated and lengthy process [7,8,10,11]. To avoid these complications, rather than using the ML algorithm, we shall apply the least squares (LS) approach to obtain frequency offset and channel estimation for OFDM in frequency-selective fading channels. As will be seen, straightforward solutions can be readily obtained for our estimators. For frequency synchronization, no channel knowledge is required. For channel estimation, we can directly estimate the channel frequency response (CFR) rather than channel impulse response (CIR). To our knowledge, such an LS estimation of CFR is not found elsewhere in the literature. Finally, an analysis is given on the error rate performance under the impact of imperfect estimations.

This paper is organized as follows: Section 2 presents the signal and system model. Section 3 delineates our LS formulations for frequency synchronization and channel estimation. Section 4 analyzes the estimation performance. Section 5 derives the SER for QAM transmission in OFDM over frequency-selective Rayleigh fading channels under imperfect estimations and comparison is made to the ideal SER performance under perfect estimations. Then, section 6 presents simulation results of the estimation performances as well as SER performances. Comparisons are also made with theoretical results. Finally, section 7 draws conclusions.
2. Signal and System Model

Consider a mobile radio channel. Let $\beta = \nu \cos \varphi / c$ be the Doppler shift factor with $\nu$ being the speed of the mobile station (MS), $c$ the speed of light, and $\varphi$ the angle of the MS moving direction with respect to the base station. Then, after experiencing Doppler shift, the resultant analog baseband signal from an OFDM transmitter is given by (ignoring the cyclic prefix)

$$x(t) = e^{j2\pi f_{c}t} \sum_{k=0}^{N-1} X_{k} e^{j2\pi f_{k}(1 - \beta)t}, \quad 0 < t < T$$

where $T$ is one OFDM block duration in units of seconds, $N$ is the OFDM block length in symbol units ($N$ is also the size of the discrete Fourier transform (DFT) or inverse DFT (IDFT) used in the OFDM system), $X_{k}$ is the data source symbol, visualized in the frequency domain, carried by the $k$th subcarrier of frequency $f_{k} = k\Delta f$, $k = 0, 1, \ldots, N - 1$, with $\Delta f = 1 / T$ being the spacing between subcarriers, and $f_{c}$ is the main carrier frequency. Further, let $z(t)$ be the complex baseband additive white Gaussian noise with zero mean and two-sided power spectral density $\sigma_{z}^{2}$ and $h(t)$ the baseband channel impulse response with transfer function $H(f)$. Then, the noisy received signal is

$$u(t) = x(t) * h(t) + z(t) = y(t) + z(t)$$

$$u(t) = e^{j2\pi f_{c}t} \sum_{k=0}^{N-1} X_{k} e^{j2\pi f_{k}(1 - \beta)t} * h(t) + z(t)$$

$$u(t) = e^{j2\pi f_{c}t} \sum_{k=0}^{N-1} X_{k} e^{j2\pi f_{k}(1 - \beta)t} H(f_{k} (1 + \beta)) + z(t)$$

where $*$ denotes convolution and $y(t) = x(t) * h(t)$ is the noise-free received signal. In (2), the fact $H(f_{k} (1 + \beta)) = \int_{-\infty}^{\infty} h(t) e^{j2\pi f_{k}(1 + \beta)t} dt$ has been used. Normally, $\beta$ is very small. For example, consider a mobile speed of $\nu = 100$ km/hr. Then, $|\beta| = |\nu \cos \theta / c| \leq \nu / c = 9.26 \times 10^{-3}$. Therefore, we will use the approximations $1 + \beta \approx 1$ and $H(f_{k} (1 + \beta)) \approx H(f_{k})$. Use of these approximations is equivalent to ignoring the subcarrier Doppler spreads.

The local oscillator at the receiver for demodulation is assumed to have a frequency offset $\Delta f_{r}$. Thus, the received signal after demodulation becomes $r(t) = u(t)e^{j2\pi f_{MT}t}$ which is then analog-to-digital converted into digital form to be input to the DFT as

$$r_{n} = u(t)e^{j2\pi (Nn/T + \nu_{0})t} \left|_{t=nT} \right. = \frac{1}{N} \sum_{k=0}^{N-1} H_{k} X_{k} e^{j2\pi nk/N} + w_{n}$$

$$n = 0, 1, \ldots, N - 1$$

where $\Delta t = T / N = 1 / N\Delta f$, $H_{k} = H(f_{k})$, $w_{n} = z(t)e^{j2\pi nk/N}$, and $\nu = \Delta f / \Delta f$. Note that $\{w_{n}\}$ are independent, identically distributed (i.i.d.) complex Gaussian random variable (RV) with zero mean and variance $\sigma_{z}^{2}$ and we shall use the familiar notation $w_{n} \sim N(0, \sigma_{z}^{2})$.

For frequency-selective channels, the discrete-time CFR is expressed as

$$H_{k} = \sum_{n=0}^{N-1} h_{n} e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N - 1$$

where we assume the channel is causal and its length spans $\nu$ data samples. The discrete-time baseband CIRs $\{h_{n}\}$ are spatially uncorrelated. We also assume that each $h_{n}$ (and hence each $H_{k}$) will remain constant over many OFDM blocks (slow quasi-static fading). This implies that the maximum Doppler frequency must satisfy $\nu T = \nu f_{c} \ll 1 / T$ corresponding to a mobile speed $\nu \ll c / f_{c}(T)$. Using the 802.11a standard with $f_{c} = 5$ GHz and $\Delta f = 1 / T = 312.5$ kHz, this requires $\nu \ll 67,500$ km/hr, hence $|\beta| \ll \nu / c < 6.25 \times 10^{-5}$. Apparently, this requirement for slow quasi-static fading is easily met in practice. For example, if $\nu = 60$ km/hr, then $f_{MT}T \approx 10^{-3}$ and it is reasonable to assume fading to remain unchanged over several hundred OFDM blocks.

Now, the DFT output at the receiver can be computed as

$$R_{k} = \sum_{n=0}^{N-1} r_{n} e^{-j2\pi nk/N}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi \nu_{0}(f_{MT} + c)N} \frac{1}{N} \sum_{p=0}^{N-1} H_{p} X_{p} e^{j2\pi np/N} e^{-j2\pi nk/N} + W_{k}$$

$$k = 0, 1, \ldots, N - 1$$

where $W_{k} = \sum_{n=0}^{N-1} w_{n} e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} z_{n} e^{-j2\pi (k-c)/N}$. Note that...
\( \{z_n\} \) are i.i.d. complex Gaussian RVs with zero mean and variance \( \sigma_z^2 \). So \( W_i \sim N(0, N\sigma_z^2) \) regardless of what value \( \varepsilon \) takes. We shall hereon use \( \sigma_z^2 \). It can be easily seen that \( \{W_i\} \) are also i.i.d.

In view of (3) and (5), it should suffice to estimate the combined frequency offset \( \delta = \beta_f T + \varepsilon \) to correct Doppler spread and carrier mismatch. As demonstrated earlier, the magnitude of \( \beta_f T \) is usually very small. On the other hand, the magnitude of \( \varepsilon \) could be small or large. We shall assume an initial frequency acquisition has been performed so that \( |\varepsilon| < 0.5 \). Thus, maximum frequency offset will be within half the subcarrier spacing so that no subcarrier will be shifted to neighboring subcarrier bands.

### 3. The Least Squares Approach for Frequency Synchronization and Channel Estimation

We first develop the scheme of estimating the combined frequency offset \( \delta \). We shall use \( P_k \) to denote the training or pilot symbols to distinguish it from the transmitted data symbol \( X_k \). Suppose, before transmission, we append \( L \) additional blocks of sequence identical to the intended transmitted block which is called the initial block or 0th block. With this arrangement, the demodulated output at the receiver can be written as (breaking the sequences block by block with \( \{r_n\} \) given by (3) as block 0 and re-denoted as \( \{\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{L-1}\} \), where \( X_i \) is replaced by \( P_k \))

\[
\tilde{r}_{i,n} = r_{i,n+1} = \tilde{y}_{i,n} + \tilde{w}_{i,n}, \quad l = 0, 1, \ldots, L, \quad n = 0, 1, \ldots, N-1
\]  

where \( \tilde{w}_{i,n} = w_{i,n+1} = z_{i,n+1} e^{j2\pi(N+n+1)/N} \) is a Gaussian RV having zero mean and variance \( \sigma_z^2 \), and

\[
\tilde{y}_{i,n} = y_{i,n+1} = y(T)_{i=(n+1)} + \sum_{k=0}^{N-1} H_i P_k e^{j2\pi(N+n+1)/N} \tag{7}
\]

In (7), we have assumed that the channel fading remains fixed at least over the \( (L+1) \) blocks as was justified earlier for normal Doppler spread values, provided \( L \) is not too large. By defining the vectors \( \tilde{r}_i = [\tilde{r}_{i,0}, \tilde{r}_{i,1}, \ldots, \tilde{r}_{i,L-1}]^T \), \( \tilde{y}_i = [\tilde{y}_{i,0}, \tilde{y}_{i,1}, \ldots, \tilde{y}_{i,L-1}]^T \), and \( \tilde{w}_i = [\tilde{w}_{i,0}, \tilde{w}_{i,1}, \ldots, \tilde{w}_{i,L-1}]^T \) with \( T \) denoting transpose and using (7), (6) can be re-written in vector form as

\[
\tilde{r}_i = \tilde{y}_i + \tilde{w}_i = \tilde{y}_i + \tilde{w}_i e^{j2\pi l + \varepsilon l} = (\tilde{r}_{i-1} - \tilde{w}_{i-1}) e^{j2\pi l + \varepsilon l} + \tilde{w}_i e^{j2\pi l + \varepsilon l} = \tilde{r}_{i-1} - \tilde{w}_{i-1} e^{j2\pi l + \varepsilon l} + \tilde{w}_i e^{j2\pi l + \varepsilon l} \tag{8}
\]

It is easy to see that the combined noise vector \( \varepsilon = -\tilde{w}_{i-1} e^{j2\pi l + \varepsilon l} + \tilde{w}_i \) is a Gaussian random vector with mean 0 and variance \( 2\sigma_z^2 \). We further stack the vectors of (9) to obtain

\[
\begin{bmatrix}
\tilde{r}_0 \\
\tilde{r}_1 \\
\vdots \\
\tilde{r}_{L-1}
\end{bmatrix} =
\begin{bmatrix}
\tilde{r}_0 \\
\tilde{r}_1 \\
\vdots \\
\tilde{r}_{L-1}
\end{bmatrix} e^{j2\pi l + \varepsilon l} +
\begin{bmatrix}
\varepsilon_0 \\
\varepsilon_1 \\
\vdots \\
\varepsilon_{L-1}
\end{bmatrix}
\tag{9}
\]

Define the three \( N(L+1) \times 1 \) vectors \( r_f = [\tilde{r}_0^T, \tilde{r}_1^T, \ldots, \tilde{r}_{L-1}^T] \), \( r_g = [\tilde{r}_0^T, \tilde{r}_1^T, \ldots, \tilde{r}_{L-1}^T]^T \), and \( e = [\varepsilon_0^T, \varepsilon_1^T, \ldots, \varepsilon_{L-1}^T]^T \). Equation (9) can then be rewritten as

\[
r_f e^{j2\pi l} = r_f - e
\]  

We can view (10) as an over-determined system of complex linear simultaneous equations in one variable \( x = e^{j2\pi l} \). Then, we can apply the method of least squares to obtain

\[
\hat{x} = (r_f^H r_f)^{-1} r_f^H r_f = \frac{\sum_{l=0}^{L-1} r_l^H r_l}{\|r_f\|^2} \tag{11}
\]

where the superscript \( H \) denotes Hermitian transpose. It is well known from parameter estimation theory that, when the error \( e \) is Gaussian, the ML estimate and LS estimate will yield the same result if the same samples are used [12]. This can indeed be readily shown for the estimator of (11). Thus, by applying the invariance property of the ML estimator [12], we can readily find

\[
\hat{\delta} = \frac{1}{2\pi} \tan^{-1} \left[ \frac{\text{Im}(\sum_{l=0}^{L-1} r_l^H \tilde{r}_f)}{\text{Re}(\sum_{l=0}^{L-1} r_l^H \tilde{r}_f)} \right] \tag{12}
\]

where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) respectively mean the real and
imaginary part.

With $\delta$, we correct the offset by multiplying (6) by $e^{-j2\pi(\alpha_0+\Delta\delta)k/N}$ to obtain $r'_{\alpha_0} = \tilde{r}_\alpha e^{-j2\pi(\alpha_0+\Delta\delta)k/N}$ (before $\tilde{r}_\alpha$ is fed to the DFT). If the estimate of good accuracy, the estimate error $\Delta\delta$ will be very small. Then, after frequency offset correction, the $k$th subcarrier DFT output (during training mode) can be approximated as

$$R'_k = e^{j2\pi\Delta\delta} \sum_{n=0}^{N-1} e^{j\frac{2\pi n k}{N}} \frac{1}{N} \left( \sum_{p=0}^{N-1} H_p P_p e^{j\frac{2\pi n p}{N}} \right) W'_k + W'_k, = e^{j2\pi\Delta\delta} \sum_{n=0}^{N-1} e^{j\frac{2\pi n k}{N}} \frac{1}{N} \left( \sum_{p=0}^{N-1} H_p P_p e^{j\frac{2\pi n p}{N}} \right)$$

$$+ W'_k \approx H_k P_k e^{j\frac{2\pi (N\alpha_0+\Delta\delta)}{N \lambda}} + W'_k \approx H_k P_k (1 + j\alpha_0 \Delta\delta) + W'_k,$$

$l = 0, 1, \ldots, L$

(13)

where $\alpha_0 = \pi(2N + N - 1) / N$, $W'_k = \sum_{n=0}^{N-1} W_n e^{-j2\pi(\alpha_0+\Delta\delta)N} e^{-j2\pi n k} N \sim N(0, \sigma^2)$, and we have used the approximations $e^{j\theta} \approx 1 + j\theta$, $\sin[\pi\Delta\delta / N] \approx \sin[\pi\Delta\delta] / N$, and $p - k + \Delta\delta \approx p - k$.

If the frequency offset correction were perfect (offset-free), then $\Delta\delta = 0$ and (13) would reduce to

$$\bar{R}'_k = H_k P_k + W'_k, l = 0, 1, \ldots, L$$

(14)

where $\bar{R}'_k$ stands for the offset-free DFT output. Equations (14) form a system of $(L+1)$ simultaneous linear equations for one single variable $H_k$ and we can thus easily apply the method of LS to get the channel estimators as

$$\hat{H}_k = (P'_k P_k)^{-1} P'_k \bar{R}'_k = \sum_{l=0}^{L} R'_k \left( \begin{array}{c} \frac{1}{L+1} \end{array} \right)$$

(15)

where $P'_k = P_k \{1, 1, \ldots, 1\}^T$ is an $(L+1)$ vector and $\bar{R}'_k = \{\bar{R}'_0, \bar{R}'_1, \ldots, \bar{R}'_L\}^T$.

In reality, perfect frequency offset correction is not very likely to happen. That is, it is not very likely to obtain the offset-free outputs $\{\bar{R}'_k\}$ of (14). One can only use the $\{R'_k\}$ of (13). Thus, in practice, we shall take

$$\hat{H}_k = \left( \sum_{l=0}^{L} R'_k \right) (L+1) P'_k, k = 0, 1, \ldots, N-1$$

(16)

Equations (12) and (16) are respectively our estimators of OFDM frequency offset and channel frequency response for frequency-selective channels. A processing flow chart of the proposed scheme is depicted in Figure 1.

4. Analysis of Estimation Performance

4.1 Frequency Offset Estimation Performance

From (12), we see that $\sum_{l=1}^{L} R'_{k,l} = \sum_{l=1}^{L} \bar{r}'_{k,l} e^{j2\pi k l/N}$.

Then, using the fact that $\bar{y}_l = \bar{y}_0 e^{j2\pi k l/N}$ and following Moose [2], we can readily show that, for high signal-to-noise ratio (SNR) and small estimate error $\Delta\delta = \delta - \hat{\delta}$,

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\[
\hat{\delta} - \delta = \frac{1}{2\pi L} \text{Im} \left[ \frac{1}{\| \tilde{y}_0 \|^2} \tilde{y}_0^H \hat{w}_e e^{j2\pi \delta} + \hat{w}_o \right]
\] (17)

Note that \( \tilde{y}_0 = [\tilde{y}_{0,0}, \tilde{y}_{0,1}, ..., \tilde{y}_{0,N-1}]^T \) is the noise-free received signal vector of the initial training block, with \( \tilde{y}_{0,n} = e^{j2\pi nk/N} \sum_{n=0}^{N-1} H_k P_k e^{j2\pi nk/N}, n = 0, 1, ..., N-1 \) (see (7)). For given \( \tilde{y}_0 \) and \( \delta \), we easily see that the conditional mean \( E[\Delta \delta | \tilde{y}_0, \delta] = E[\hat{\delta} - \delta | \tilde{y}_0, \delta] = 0 \). Hence, for small errors at high SNR, the frequency offset estimate is unbiased. Using (17), the conditional variance of the estimate can be evaluated as

\[
E[(\hat{\delta} - \delta)^2 | \tilde{y}_0, \delta] = \frac{\sigma_w^2}{4\pi^2 L \| \tilde{y}_0 \|^2} = \frac{\sigma_w^2}{4\pi^2 L N \gamma}
\] (18)

Note that the block signal power \( \| \tilde{y}_0 \|^2 \) would increase with \( N \). We have thus defined the block SNR as \( \gamma = \| \tilde{y}_0 \|^2 / \sigma_w^2 \). We can see that, as \( L \) and/or \( N \) are increased, the estimate gets better. For \( N = 256 \), two repeated training blocks as treated by Moose (\( L = 1 \)) prove to be sufficient for accurate offset estimates [2].

### 4.2 Channel Estimation Performance

To evaluate the channel estimation performance, we will substitute (13) into (16) to get

\[
\hat{H}_k = H_k + \frac{1}{(L+1)} \sum_{n=0}^{L} W_{P_k} \] (19)

whence, for given \( H_k \) and \( \delta \), the conditional mean of \( \hat{H}_k \) is

\[
E[\hat{H}_k | H_k, \delta] = H_k
\] (20)

where we have used \( E[\Delta \delta | \tilde{y}_0, \delta] = 0 \). Hence, for small offset errors at high SNR, the channel estimate is unbiased. Then, using (17), (18), and (19), we can readily calculate the conditional channel estimate variance (at given \( H_k \) and \( \delta \) as

\[
E[(\hat{H}_k - H_k)^2 | H_k, \delta] = C \frac{\sigma_{w}^2}{\| \tilde{y}_0 \|^2} + \frac{\sigma_w^2}{(L+1) \| P_k \|^2}
\] (21)

where

\[
C = \frac{(NL + N - 1)}{2N^2 L} \left[ \frac{(NL + N - 1)}{2NL} + \frac{1 - \cos(2\pi L \delta)}{L+1} \right]
\] (22)

The overall average variance is defined as

\[
\frac{1}{L} E[\| \Delta H_k \|^2 | H_k, \delta] = \frac{1}{L} \sum_{n=0}^{N-1} E[\| \hat{H}_k - H_k \|^2 | H_k, \delta]
\] (23)

Equations (20) and (21) through (23) give the theoretical mean and variance of the channel estimates when the frequency offset estimate is imperfect. From (21), we see that, with a given training sequence \( \{P_k\} \), the channel estimate variances will be different for different subcarriers due to the presence of \( H_k \).

### 5. QAM SER Performance

We first present the SER performance of OFDM systems over frequency-selective fading channels under the scenario of imperfect estimations. Then, to see the impact of imperfect estimations, we also derive the ideal SER when the estimates are perfect. We shall use square QAM transmission for the analysis.

#### 5.1 Performance under Imperfect Estimations

After the \( \delta \) correction and channel estimation in training mode, we switch to data mode. The received DFT output of the \( k \)th subcarrier is given by

\[
R^\delta_k = \frac{1}{N} \sum_{n=0}^{N-1} H_{p_k} X_k e^{j2\pi n(k-\delta)/N} + W_k^\delta
\] (24)

where we have resumed the use of \( \{X_k\} \) for transmitted data symbols and the superscript \( \# \) is used to denote data mode. Assuming \( \hat{H}_k \) is the channel estimate for a given offset \( \delta \) and given channel realizations \( \{H_k\} \), then the data estimate is given by

\[
\hat{X}_k = \frac{R^\delta_k}{H_k} = \frac{B H_k X_k}{H_k} + \frac{1}{N H_k} \sum_{n=0}^{N-1} H_{p_k} X_k e^{j2\pi (p-k-\delta)/N} + W_k^\delta
\]

\[
= X_k + I_k + W_k^\delta, \quad k = 0, 1, ..., N-1
\] (25)
where $\beta = \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi in/N}$ and $W_k' = W_k' / \hat{H}_k$. Note that

$$\frac{\beta H_k X_k}{\hat{H}_k}$$

is the desired term but with a bias $\mu_k = \frac{\beta H_k X_k}{\hat{H}_k}$.

$X_k$ and $I_k = \frac{1}{N\hat{H}_k} \sum_{n=0}^{N-1} H_{np} X_p e^{2\pi i (n-p-k)/N}$ is the ICI.

Consider fixed channel realizations $\{H_k\}$ and a given estimate pair of $(\Delta \delta, \hat{H}_k)$. Then when $N$ is large, the ICI term will consist of the sum of a large number of independent RVs $\{X_p, p \neq k\}$. We can thus invoke the central limit theorem and approximate the ICI plus noise terms as a Gaussian RV [13] with zero mean and variance given by

$$\sigma^2 = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} (H_{np} X_p e^{2\pi i (n-p-k)/N} = \frac{\sigma^2}{N}$$

where $\sigma^2 \Delta = E[|X_k|^2]$ is the average transmitted data power. Thus for fixed channel realizations $\{H_k\}$ and a given estimate pair $(\Delta \delta, \hat{H}_k)$, the carrier-to-ICI plus noise ratio (CINR) is $\sigma^2 / \sigma^2$.

Now for square $M$-QAM transmission, the two quadrature $\sqrt{M}$-PAM components of the data symbol $X_k$ are $X_{kc}$ and $X_{ks}$, each randomly taking on the value from the sets $\{A = (2m+1 + \sqrt{M})d, m = 1, 2, ..., \sqrt{M}\}$, where $d$ is a constant used for power control. For fixed channel realizations $\{H_k\}$ and a given estimate pair $(\Delta \delta, \hat{H}_k)$, we can readily follow the standard SER computation procedure for the square QAM [14]. For our case here, $\mu_k$ is a function of the data symbol $X_k$. We will thus write $\mu_k(X_k)$ from here on. We first compute the SER conditioned on $(\Delta \delta, \hat{H}_k, \hat{H}_k)$ at each $X_k$, then average over $\{X_k\}$. Thus, the conditional SER given $(\Delta \delta, \hat{H}_k, \hat{H}_k)$ for the $k$th subcarrier is

$$P_{SA,k}(e|\Delta \delta, H_k, \hat{H}_k) = \frac{1}{M} \sum_{X_k} \left[ 1 - P_{SM,k}(e) - P_{SM,k}(X_k) \right]$$

where $P_{SM,k}$ and $P_{SM,k}$ are the two conditional quadrature $\sqrt{M}$-PAM SERs at $X_k$. Three possible situations need be considered, viz., 1) $X_k$ at one of the four corner points of the square $M$-QAM constellation; 2) $X_k$ at one of the $4(\sqrt{M} - 2)$ border points excluding corner points; 3) $X_k$ at one of the inner points. The calculations are straightforward. We present the results as follows:

$$P_{SA,k} = \frac{Q \left( d + \text{sign}(\text{Re}[X_k]) \cdot \text{Re}[\mu_k(X_k)] \right)}{\sigma_k / \sqrt{2}}$$

$$\text{sign}(\cdot)$$

is the familiar sign function and

$$f(x) = 0 \text{ when } x = \pm (\sqrt{M} - 1)d \text{ (boundary point)}$$

$$f(x) = 1 \text{ when } x = \pm (\sqrt{M} - 1)d \text{ (non-boundary point)}$$

We can now substitute (28) into (27) to obtain the conditional SER.

Next, consider frequency-selective Rayleigh fading channels. Using (4), we see that $H_k$ has a probability density function (PDF) given by

$$p(H_k) = \frac{1}{\pi \sigma^2_{H_k}} \exp \left( -\frac{|H_k|^2}{\sigma^2_{H_k}} \right), \quad k = 1, ..., N - 1$$

where $\sigma^2_{H_k} = \sum_{m=0}^{M-1} \sigma^2_{H_m}$ which is independent of $k$, and $\sigma^2 = E[|h_m|^2]$. Note that for fixed channel realizations and large $N$, we can make the approximation $\Delta \delta \sim N(0, \sigma^2_{H_k})$ with $\sigma^2_{H_k}$ given by (18). Then from (19), the RV $H_k$ is a function of both $\Delta \delta$ and $W_{\ell,k}$, where $W_{\ell,k} = \sum_{j=0}^{N-1} W_{\ell,j} \sim N(0, (L + 1) \sigma^2)$. Therefore, the long-term average SER for the $k$th subcarrier over fading is calculated as

$$P_{SA,k} = \prod_{i=1}^{L} \left[ P_{SA,k}(e|\Delta \delta, H_k, \hat{H}_k)p(H_k)\right]$$

$$\text{where } p_{\Delta \delta}(\Delta \delta) \text{ and } p_{W_{\ell,k}}(W_{\ell,k}) \text{ are respectively the PDF}$$
of $\Delta \delta$ and $W_{T_k}$. Note that both $H_k$ and $W_{T_k}$ are complex. The average SER of (31) can be obtained by numerical simulation. To perform the simulation, a random call is made of the channel realizations $\{H_k\}$ with $\delta$ given. Then, the estimates $\hat{\delta}$ and $\hat{H}_k$ corresponding to $\{H_k\}$ are respectively obtained from (12) and (16) during training. Substitute the set $(\Delta \delta, H_k, \hat{H}_k)$ into (27) to find $P_{M,k}(e \mid \Delta \delta, H_k, \hat{H}_k)$. The same procedure is repeated for sufficient number of times (sufficient channel realizations). Summing all the $\{P_{M,k}(e \mid \Delta \delta, H_k, \hat{H}_k)\}$ thus obtained and dividing by the number of repetitions, we can get the average SER $P_{M,k}$. Note that, only one estimate pair $(\hat{\delta}, \hat{H}_k)$ is used for a given set of realizations $\{H_k\}$ in simulations. This is because, if the experiment is repeated sufficient number of times, a stochastic averaging effect will take place. In our simulations, $2 \times 10^5$ channel realizations are used. The same simulation technique as described above has also been adopted in [15] for single carrier frequency domain equalizers in frequency-selective slow fading channels and in [16] for OFDM systems in frequency-selective fast fading channels. We note here that the above technique can also be readily extended to Rician fading.

The overall system average SER over fading is simply obtained as

$$P_M = \frac{1}{N} \sum_{k=0}^{N-1} P_{M,k}$$

(32)

5.2 QAM SER with Perfect Estimates

If all estimates are perfect (i.e., $\Delta \delta = 0$ and $\hat{H}_k = H_k$), there will be no ICI and $\mu_k = 0$ (Since $\beta = 1$). Then, (25) simply becomes

$$\hat{X}_k = X_k + \frac{W_k^*}{H_k}$$

(33)

Note that $W_k^* \sim N(0, \sigma_w^2)$. Now, (27) should be modified to

$$P_{M,k}(e \mid H_k) = 1 - [1 - P_{\Delta \delta,k}(e \mid H_k)]^2$$

(34)

Here, we use

$$P_{\Delta \delta,k}(e \mid H_k) = P_{\Delta \delta,k}(e \mid H_k) = P_{\Delta \delta,k}(e \mid H_k) = \frac{2(\sqrt{M} - 1)}{\sigma_w} \left[ \frac{d}{\sigma_w} \right] = 2(\sqrt{M} - 1)Q\left( \frac{3\gamma_k}{\sqrt{M} - 1} \right)$$

(35)

where $\gamma_k = \frac{1}{\sigma_H^2} \frac{\sigma_x^2}{\sigma_w^2}$ is the average received signal-to-noise ratio (SNR) for the $k$th subcarrier with a fixed $H_k$ and the fact that $\sigma_x^2 = 2(\sqrt{M} - 1)d^2/3$ has been used. For Rayleigh fading, it is well known that $p_{\gamma_k} = \gamma_k \sigma_x^2 / \sigma_w^2 > 0$ has an exponential PDF given by $p_{\gamma_k} = \frac{1}{\sigma_{\gamma_k}} e^{-\gamma_k/\sigma_{\gamma_k}^2}$.

Using the variable transformation $\gamma_k = \gamma_k \sigma_x^2 / \sigma_w^2$, we readily obtain the PDF of $\gamma_k$ as

$$p_{\gamma_k}(\gamma_k) = \frac{\sigma_{\gamma_k}^2}{\sigma_{\gamma_k}^2} e^{-\gamma_k/\sigma_{\gamma_k}^2}, \quad \gamma_k > 0$$

(36)

The average SER of the $k$th subcarrier can now be easily calculated using the moment generating function technique [17]. Using (35) and (36), the result can be obtained in a closed-form as

$$P_{M,k} = \int_{0}^{\infty} \frac{2(\sqrt{M} - 1)}{\sigma_w} \left[ \frac{3\gamma_k}{\sigma_H^2} \right] e^{-\gamma_k/\sigma_{\gamma_k}^2} d\gamma_k$$

$$= \frac{2}{\sqrt{M} - 1} \frac{2a_k(\sqrt{M} - 1)}{\sigma_{\gamma_k}} \left[ \frac{\pi \sqrt{1 + 2a_k^2}}{\arctan \sqrt{1 + 2a_k^2}} \right]$$

(37)

where $a_k = \frac{\sqrt{3\gamma_k}}{2(\sqrt{M} - 1)}$ and $\sigma_{\gamma_k} = \sigma_{\gamma_k}^2 / \sigma^2_w$ is the average received SNR of the $k$th subcarrier for the fading channels. The overall system average SER under perfect estimations is obtained by substituting (37) into (32). In the next section, SER performances both under perfect and imperfect estimations will be compared.

6. Simulation Results

In this section, we shall present simulation results of our estimation performance as well as the SER performance in frequency-selective Rayleigh fading channels. All simulation results will be compared with theoretical predictions.

Figure 2 shows the frequency offset variance $E[(\delta - \delta)^2]...
As a function of $\gamma$ for three values of $L = 1, 3, 13$ at fixed channel realizations of $\{H_k\}$ as obtained from Monte Carlo simulations as well as from the theoretical result of (18). An exponential power profile has been assumed for a frequency-selective channel with dispersion length $\nu = 16$ and unity channel power, i.e., $\sum_{n=0}^{N-1} |h_n|^2 = 1$.

We use 64-point DFT ($N = 64$) and assume an actual frequency offset of $\delta = 0.15$. From Figure 2, we see that the simulated and theoretical curves coincide well at high SNR values as expected since the theoretical result of (18) was derived based on high SNR values. We also observe that, when SNR is high up to 30 dB, $L = 1$ (two training blocks) will yield an estimate error variance about $4 \times 10^{-7}$. Therefore, use of $L > 1$ at high SNRs is unnecessary. For low SNR values, the theoretical variance of (18) is less accurate. From the figure, we see that, when SNR is as low as 10 dB, use of $L = 3$ proves to be good enough as the simulated estimate error variance there becomes as low as $5 \times 10^{-6}$, thus any higher value of $L > 3$ is unnecessary.

Next, Figure 3 gives the overall average channel estimate variance obtained from Monte Carlo simulations as well as from the theoretical results of (23) under imperfect estimations for the same three $L$ values. From the figure, we see that, if we wish the variance to go below $10^{-4}$ at 30 dB SNR, we need to use at least an $L$ value as large as 13. We also note here that the frequency offset estimation variance of (19) was derived based on high SNR and the channel estimation relies on the frequency offset estimation result. Therefore, the theoretical prediction for the channel estimation variance should have a slight discrepancy from the actual result at low SNR. In Figure 3, this discrepancy is not detectable for small $L$ values ($L = 1, 3$). But for a large $L = 13$, we can clearly see the discrepancy at low SNR.

Finally, Figure 4 presents the curves of QAM SER vs. the total average SNR $\bar{\gamma} = \frac{1}{N} \sum_{k=0}^{N-1} \bar{\gamma}_k$ for OFDM systems over frequency-selective Rayleigh fading channels using the same parameters of $\delta = 0.15$ and $N = 64$ as well as the same channel model of exponential power profile as above. We have chosen the $L = 1$ and $L = 3$ cases and SER performances under imperfect and perfect estimations are both given which are seen to be quite close to each others indicating that our estimators are indeed quite good.
7. Conclusion

By employing least squares formulation, we utilize repeated identical blocks of training data to achieve frequency synchronization and channel estimation for OFDM over frequency-selective fading channels. All simulation results agree well with theoretical predictions, proving that our algorithm indeed provides good estimators.

References


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