Natural Vibration of Visco-Elastic Plate of Varying Thickness with Thermal Effect

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Abstract

In the modern technology, the plates of variable thickness are widely used in engineering applications i.e. nuclear reactor, aeronautical field, naval structure, submarine etc. In this paper, effect of thermal gradient on vibration of square plate of varying thickness is studied. Thermal effect and thickness of plate vary bi-parabolic i.e. parabolic in x-direction and parabolic in y-direction. Rayleigh Ritz technique is used to calculate the fundamental frequencies. The frequencies corresponding to the first two modes of vibrations are obtained for a square plate for different values of taper constant and thermal gradient. Results are presented in Table and Graphical form.

Key Words: Vibration, Frequencies, Square Plate, Vibration, Thermal Gradient

1. Introduction

In the engineering we cannot move without considering the effect of vibration because almost all machines and engineering structures experiences vibrations. As technology develops new discoveries have intensified the need for solution of various problems of vibrations of plates with elastic or visco-elastic medium. Since new materials and alloys are in great use in the construction of technically designed structures therefore the application of visco-elasticity is the need of the hour.

With the advancement of technology, the requirement to know the effect of temperature on visco-elastic plates of variable thickness has become vital due to their applications in various engineering branches such as nuclear power plants, engineering, industries etc. Further in mechanical system where certain parts of machine have to operate under elevated temperature, its effect is far from negligible and obviously cause non-homogeneity in the plate material i.e. elastic constants (Young’s modulus etc.) of the materials becomes functions of space variables.

In an up-date survey of literature, authors have come across various models to account for non-homogeneity of plate materials proposed by researchers dealing with vibration but none of them consider non-homogeneity with thermal effect on visco-elastic plates. It also indicates that sufficient work on one dimensional temperature variation has been done but negligible work has been done in the field of two dimensional temperature variation.

Recently, A. K. Gupta and Anupam Khanna [1] studied the effect of linear temperature variation on vibrations of parallelogram plate of linearly varying thickness. A. K. Gupta and Anupam Khanna [2] studied the vibration of clamped visco-elastic rectangular plate with bi-parabolic thickness variations but no thermal effect has been considered. A. K. Gupta and Anupam Khanna [3] has been studied on free vibration of clamped visco-elastic rectangular plate having bi-direction exponentially thickness variations without the consideration of thermal effect. A. K. Gupta and A. Khanna [4] discussed the vibration of visco-elastic rectangular plate with bi-linearly thickness variations in both directions with no temperature effect. Anupam Khanna, Ashish Kumar

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The aim of present investigation is to study two dimensional thermal effects on the vibration of visco-elastic square plate whose thickness varies parabolic in x-direction. It is assumed that the plate is clamped on all the four edges and its temperature varies parabolic in both the directions. Due to temperature variation, we assume that non homogeneity occurs in modulus of elasticity. For various numerical values of thermal gradient and taper constants; frequency for the first two modes of vibration are calculated with the help of latest software. Authors compare the results of present paper with [15] and findings are shown in Tables 1 and 2. Graphs are also provided for the results of present paper.

2. Equation of Motion and Analysis

The governing differential equation of transverse motion of a visco-elastic plate of variable thickness in Figure 1 [14]:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = \rho h \frac{\partial^2 w}{\partial t^2}$$ (1)

The expression for $M_x$, $M_y$, $M_{xy}$ are given by

<table>
<thead>
<tr>
<th>Table 1. Frequency vs. thermal gradient</th>
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<tr>
<td>$\alpha$</td>
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<th>Table 2. Frequency vs. taper constant</th>
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<td>$\beta_1$</td>
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<tr>
<td>Mode 1</td>
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\[ M_x = -\ddot{D}D_1 \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial x \partial y} \right) \]
\[ M_y = -\ddot{D}D_1 \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x \partial y} \right) \]
\[ M_{xy} = -\ddot{D}D_1 (1-v) \frac{\partial^2 W}{\partial x \partial y} \]

where \( \ddot{D} \) is visco-elastic operator.

On substitution the values \( M_x, M_y \) and \( M_{xy} \) from equation (2) in (1) and taking \( w \), as a product of two function, equal to \( w(x, y, t) = W(x, y)T(t) \), equation (1) becomes:

\[ \left[ D_1 \left( \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + 2 \frac{\partial^2 D_1}{\partial x^2} \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial x \partial y} \right) + 2 \frac{\partial^2 D_1}{\partial y^2} \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x \partial y} \right) \right. \]
\[ \left. + \frac{\partial^4 D_1}{\partial x^4} \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial x \partial y} \right) + 2(1-v) \frac{\partial^2 D_1}{\partial x \partial y} \frac{\partial^2 W}{\partial x \partial y} \right] \rho h W = -\ddot{T} \frac{T}{D_1} \]

Here dot denote differentiation with respect to \( t \). taking both sides of equation (3) are equal to a constant \( \rho h^2 \) (square of frequency), we have

\[ \left[ D_1 (W_{xx} + 2W_{xxy} + W_{xyy}) + 2D_1 (W_{xx} + W_{xy}) \right. \]
\[ \left. + 2D_1 (W_{xx} + W_{xy}) + D_1 (W_{xx} + W_{xy}) \right] + 2(1-v)D_1 (W_{xy} + W_{xy}) - \rho h^2 \]
\[ W = 0 \]

Eq. (4) is a differential equation of transverse motion for non-homogeneous plate of variable thickness. Here, \( D_1 \) is the flexural rigidity of plate i.e.

\[ D_1 = \frac{Eh^3}{12(1-v^2)} \]

and corresponding two-term deflection function is taken as [5]

\[ W = \frac{\left( x/a \right) \left( y/a \right) (1-x/a)(1-y/a)}{(1-x/a)(1-y/a)} \]

where \( A_1 \) and \( A_2 \) are constants to satisfy boundary conditions.

Assuming that the square plate of engineering material has a steady two dimensional parabolic temperature distribution i.e.

\[ \tau = \tau_0 (1-x^2/a^2)(1-y^2/a^2) \]

where \( \tau \) denotes the temperature excess above the reference temperature at any point on the plate and \( \tau_0 \) denotes the temperature at any point on the boundary of plate and “a” is the length of a side of square plate. The temperature dependence of the modulus of elasticity for most of engineering materials can be expressed in this

\[ E = E_0 (1-\gamma \tau) \]

where, \( E_0 \) is the value of the Young’s modulus at reference temperature i.e. \( \tau = 0 \) and \( \gamma \) is the slope of the variation of \( E \) with \( \tau \). The modulus variation (5) become

\[ E = E_0 [1-\alpha(1-x^2/a^2)(1-y^2/a^2)] \]

where \( \alpha = \gamma \tau_0 (0 \leq \alpha < 1) \), thermal gradient.

It is assumed that thickness also varies parabolic in \( x \)-directions as shown below and in Figure 2:

\[ h = h_0 (1+\beta_1 x^2/a^2) \]

where \( \beta_1 \) is taper parameters in \( x \)-directions respectively and \( h = h_0 \) at \( x = y = 0 \).

Figure 1. Square plate with bending moments.

Figure 2. Plate with parabolic varying thickness.
Put the value of $E$ & $h$ from equation (9) & (10) in the equation (5), one obtain

$$D_1 = \left[ E_0 (1 - \alpha (1 - X^2 / a^2) (1 - Y^2 / a^2) \right] \left[ (1 + \beta_s X^2 / a^2)^{1/2} \right] / 12(1 - \nu^2)$$ (11)

Rayleigh-Ritz technique is applied to solve the frequency equation. In this method, one requires maximum strain energy must be equal to the maximum kinetic energy. So it is necessary for the problem under consideration that

$$\delta (V^* - T^*) = 0$$ (12)

for arbitrary variations of $W$ satisfying relevant geometrical boundary conditions.

Since the plate is assumed as clamped at all the four edges, so the boundary conditions are

$$W = W_{xx} = 0, x = 0, a$$
$$W = W_{yy} = 0, y = 0, a$$ (13)

Now assuming the non-dimensional variables as

$$X = x / a, Y = y / a, \overline{W} = W / a, \overline{h} = h / a$$ (14)

The kinetic energy $T^*$ and strain energy $V^*$ are [2]

$$T^* = (1/2) \rho \overline{h}^2 \overline{a}^3 \int_0^1 \int_0^1 [(1 + \beta_s X^2) \overline{W}^2] dYdX$$ (15)

and

$$V^* = Q \int_0^1 \int_0^1 [1 - \alpha (1 - X^2) (1 - Y^2)] \left[ (1 + \beta_s X^2)^{1/2} \right] \left[ (\overline{W}_{xx})^2 + 2v \overline{W}_{xx} \overline{W}_{yy} + 2(1 - \nu) (\overline{W}_{xy})^2 \right] dYdX$$ (16)

where, $Q = E_0 \overline{h}^2 a^3 / 24 (1 - \nu^2)$

Using equations (15) & (16) in equation (12), one get

$$(V^* - \lambda^2 T^*) = 0$$ (17)

where

$$V^* = \int_0^1 \int_0^1 [1 - \alpha (1 - X^2) (1 - Y^2)] \left[ (1 + \beta_s X^2)^{1/2} \right] \left[ (\overline{W}_{xx})^2 + 2v \overline{W}_{xx} \overline{W}_{yy} + 2(1 - \nu) (\overline{W}_{xy})^2 \right] dYdX$$ (18)

and

$$T^* = \int_0^1 \int_0^1 [(1 + \beta_s X^2) \overline{W}^2] dYdX$$ (19)

Here, $\lambda^2 = 12 \rho \overline{h}^2 \overline{a}^3 / E_0 \overline{h}^2$ is a frequency parameter.

Equation (19) consists two unknown constants i.e. $A_1$ & $A_2$ arising due to the substitution of $W$. These two constants are to be determined as follows

$$(\partial (V^* - \lambda^2 T^*) / \partial A_n) = 0, n = 1, 2$$ (20)

On simplifying (20), we get

$$b_{n_1} A_1 + b_{n_2} A_2 = 0, n = 1, 2$$ (21)

where $b_{n_1}, b_{n_2} (n = 1, 2)$ involve parametric constant and the frequency parameter.

For a non-trivial solution, the determinant of the coefficient of equation (21) must be zero. So one gets, the frequency equation as

$$\left| \begin{array}{cc} b_{n_1} & A_1 \\ b_{n_2} & A_2 \end{array} \right| = 0$$ (22)

With the help of equation (22), one can obtains a quadratic equation in $\lambda^2$ from which the two values of $\lambda^2$ can found. These two values represent the two modes of vibration of frequency i.e. $\lambda_1$ (Mode 1) & $\lambda_2$ (Mode 2) for different values of taper constant and thermal gradient for a clamped plate.

3. Result and Discussion

All calculations are carried out with the help of latest Matrix Laboratory computer software. Computation has been done for frequency of visco-elastic square plate for different values of taper constants $\beta_1$ and thermal gradient ($\alpha$), at different points for first two modes of vibrations have been calculated numerically.

In Table 1 and Figure 3: It is clearly seen that value of frequency continuously decreases as value of thermal gradient increases from 0.0 to 1.0 for different values of taper parameters i.e. $\beta_1 = 0.4$ and $\beta_1 = 0.6$ for both the modes of vibrations.

Actually, frequency and Young’s modulus ($E$) are di-
rectly proportional to each other i.e. frequency \( \propto \sqrt{E} \). It indicates that on increasing the value of Young’s modulus (E), frequency will also increase. Using the value of E from equation (9), above relation becomes: Frequency \( \propto \sqrt{E_0 \left\{ 1 - \alpha (1 - x^2/a^2)(1 - y^2/a^2) \right\}} \) which simply shows that if thermal gradient increases, frequency decreases. Hence, numeric results of frequency in Table 1 are authentic and acceptable.

In Table 2 and Figure 4: Also, it is evident that value of frequency continuously increases as value of taper parameter increases from 0.0 to 1.0 for different values thermal gradient i.e. \( \alpha = 0.4 \) and \( \alpha = 0.6 \) for both the modes of vibrations.

From equation (19), it is clearly seen that Kinetic energy (K. E.) of plate material directly depend on taper parameter (\( \beta_1 \)) which shows that as \( \beta_1 \) increases, K. E. will also increase.

Due to this increment in K. E., molecules of materials get extra energy and their velocity increases. Due to this, their rate of oscillation/vibration i.e. frequency increases. So, one can conclude that as taper parameter increase, frequency also increases. Hence, numeric results of Table 2 are also authentic and acceptable.

4. Conclusion

Authors compare the results of present paper with [15] and placed them (in italic and bold) just below the values of frequency in Tables 1 and 2. On comparing the present results with [15], the authors concluded that values of frequencies in present paper decrease for both the modes of vibrations for the corresponding values of taper parameter i.e. \( \beta_1 = 0.4 \) & \( \beta_1 = 0.6 \) as thermal gradient increases.

In Tables 1 and 2, it is also interesting to note that difference in Mode 1 is quite more as compared to Mode 2. Hence, the main aim for our research is to develop a theoretical mathematical model for scientists and design engineers so that they can make a use of it with a practical approach, for the welfare of the human beings as well as for the advancement of technology.

References


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