Controlling Lost Sales Rate for Inventory System with Partial Backlogging Defective Items

Hsien-Jen Lin

Department of Applied Mathematics, Aletheia University,
Tamsui, Taiwan 251, R.O.C.

Abstract

We study a continuous review inventory model which involves controllable lead time, partial backlogging defective items and investment to reduce lost sales rate. Buyer’s order quantity, reorder point, lost sales rate and lead time are taken as decision variables. By framing the model, we observe that a significant amount of savings can be easily achieved to increase the competitive edge. Both normal distribution and distribution free models are discussed sufficiently. Effects of investing in lost sales rate reduction are clearly stated and savings are achieved in a numerical example. We compare the lost sales rate reduction model with fixed lost sales rate model. Also, we give the improvement of the cost performance of the distribution free approach in another numerical example.

Key Words: Inventory, Lost Sales, Defective Items, Lead Time, Minmax Distribution Free Approach

1. Introduction

More recently, most inventory problems have focused on inventory models with partial backorders. Montgomery et al. [1] are among the first who analyzed the inventory problems where, during the stock-out period, a fraction of demand is backordered and the remaining fraction is lost. The framework they proposed has encouraged many researchers to present various types of inventory models with partial lost sales. Ouyang et al. [2] and Ouyang and Wu [3] proposed a continuous review \((Q, L)\) inventory model with lead time reduction by allowing shortage with backorders. Moon and Choi [4] and Hargia and Ben-Daya [5] extended Ouyang et al.’s [2] model by allowing the reorder points as one of the decision variables and they studied the effects of lead time reduction on the modified lot size reorder point \((Q, r)\) inventory model and developed different algorithm procedures for solving these \((Q, r, L)\) models, respectively. Ouyang et al. [6] further extended the model of Moon and Choi [4] and that of Hargia and Ben-Daya [5], and permitted the setup cost as a decision variable, i.e., they established a \((Q, A, r, L)\) inventory model. We note that the underlying assumption in these papers is that the lost sales rate is prescribed to be constant and thus it is not subject to control. In real markets, many factors may affect customers’ willingness of accepting backorders during the stock-out period. Many efforts, such as upgrading the servicing facilities, maintaining a high quality of products, etc. can be made to establish a good relationship with customers and enhance customers’ loyalty, so as to motivate customers’ preference for backorders. Naturally, extra money must be spent for these efforts. As a result it is expected to have reduced cost of lost sales and reduced total expected cost. Ouyang and Chang [7] studied a continuous review \((Q, r, \alpha)\) inventory model and Ouyang et al. [8] investigated a periodic review \((T, r, \alpha, L)\) inventory model and concluded that the lost sales rate can be reduced by an effective capital investment. Later, Ouyang et al. [9] studied a periodic review \((R, T, \alpha)\) inventory model and considered two commonly used investment cost functions to reduce lost sales rate to secure more backorders. The above body of literature neglected the possible relationship between the order lot
and quality. A common unrealistic assumption of the above inventory models is that all items produced are of good quality. However, as a result of imperfect production processes of the vendor, damage in transit, or other unforeseeable circumstances, an order lot arriving at the buyer often contains defective items. These defective items will affect the on-hand inventory level, customer service level and the frequency of orders in the inventory system. So production/shipment policy determined by conventional inventory models may be inappropriate for the situation where an arrival lot contains some defective items. Therefore, it is worthwhile studying the effect of defective items on inventory problem. Since the pioneering work by Rosenblatt and Lee [10] and Porteus [11], in order to circumvent the common unrealistic assumption of good quality, many researchers have attempted to develop various imperfect-quality inventory models on this important issue. Paknejad et al. [12] presented a quality-adjusted lot-sizing model with constant lead time and stochastic demand, and considered the number of defective goods in a lot to be a random variable. Wu and Ouyang [13] incorporated the assumption of a mixture of backorders and lost sales and variable lead time into the model of Paknejad et al. [12]. There are many related studies, such as [14–25], and others.

In this paper we develop a continuous review inventory model with defective items to accommodate the real inventory systems and analyze the effects of increasing investment to reduce the lost sales rate. We formulate a continuous review inventory model for items with imperfect quality and a mixture of backorders and lost sales in which the buyer’s order quantity, reorder point, lost sales rate and lead time are decision variables. We assume that an arrival order lot may contain some defective items and the number of defective items is a random number and all defective items in each lot are assumed to be discovered and will be returned to the vendor at the time of delivery of the next lot. Thus the buyer will have two kinds of holding cost: non-defective items holding cost and defective items holding cost; an order of size is made whenever the inventory level drops to the reorder point and also, as in Liao and Shyu [26], we assumed that lead time is controllable and shortages during the lead time are allowed. The available budget of investing in lost sales reduction is limited. Models with full and partial information of lead time demand distributions are developed. We first assume that the lead time demand follows a normal distribution, and try to determine the optimal ordering policy. Then we relax this assumption by assuming that only the first two moments of the lead time demand are known. This results in the well-known min-max formulation which looks for the worst case probability distribution under the given moment constraints. The min-max problem is solved using the approach of Scarf [27] (popularized by Gallego and Moon [28]). An efficient solution algorithm is developed and numerical examples are presented to illustrate the procedures of the proposed algorithm.

The rest of this paper is organized as follows. The next section details the notation and assumptions. In section 3, we formulate the continuous review model involving controllable lead time and a random number of defective items with lost sales reduction. Both normal distribution model and distribution free model are discussed and then an algorithm is developed to find the optimal solution. Numerical examples are given to illustrate the proposed model in section 4. The final section concludes the paper.

2. Notation and Assumptions

The mathematical models of the inventory problems here are based on the following notation and assumptions.

2.1 Notation

- **A** Ordering cost per order
- **D** Expected demand per unit time (for non-defective items)
- **E[.]** Mathematical expectation
- **G** Available budget of investing in lost sales reduction
- **g(p)** The probability density function of *p*
- **h** Holding cost per non-defective item per unit time
- **h’** Holding cost per defective item per unit time, *h’ < h*
- **l(α)** Investment required to reduce the lost sales fraction from *α₀* to α
- **i** Fractional opportunity cost of capital per unit time
- **L** Length of lead time (decision variable)
- **p** Defective rate in an order lot, *p ∈ [0, 1]*, a random variable
- **Q** Order quantity including defective items (decision variable)
### 2.2 Assumptions

1. An arrival lot may contain some defective items. We assume that the number of defective items in an arriving order of size $Q$ is a binomial random variable with parameters $Q$ and $p$, where $p (0 \leq p \leq 1)$ represents the defective rate in an order lot.

2. Upon an arrival order lot $Q$ with a defective rate $p$, the entire items are inspected and all defective items are assumed to be discovered and removed from order quantity $Q$ and defective items in each lot will be returned to the vendor at the time of delivery of the next lot.

3. Inspection is nondestructive and error free.

4. Inventory is continuously reviewed and replenishments are made whenever the inventory level (based on the number of non-defective items) drops to the reorder point $r$.

5. The reorder point $r = \text{expected demand during lead time + safety stock (SS)}$, and $SS = k \times \text{(standard deviation of lead time demand)}$, that is, $r = DL + k \sigma \sqrt{L}$, where $k$ is the safety factor.

6. The lead time $L$ has $m$ mutually independent components. The $i$th component has a minimum duration $a_i$ and normal duration $b_i$, and the crashing cost per unit time $c_i$. Furthermore, these $c_i$ are assumed to be arranged such that $c_1 \leq c_2 \leq \ldots \leq c_m$.

7. The components of lead time are crashed one at a time starting with the component of least $c_i$ and so on.

8. Let $L_0 = \sum_{j=1}^{m} b_j$ and $L_i$ be the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then $L_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1, 2, \ldots, m$ and the lead time crashing cost per cycle $C(L)$ for a given $L \in (L_0, \ldots, L_{m-1}]$ is given by $C(L) = c(L_{m-1} - L) + \sum_{j=1}^{m} c_j (b_j - a_j)$.

9. The lost sales fraction $\alpha$ can be reduced by capital investment $I(\alpha)$ and $I(\alpha)$ follows a logarithmic function given by $I(\alpha) = \frac{1}{\delta} \ln \left( \frac{\alpha}{\alpha_0} \right)$ for $0 < \alpha \leq \alpha_0$, where $\delta$ is the percentage decrease in $\alpha$ per dollar increase in $I(\alpha)$, and $\alpha_0$ is the original fraction of the shortage that will be lost. This function is consistent with the Japanese experience as reported in Hall [29], and has been utilized in literature to formulate various investing options (see, e.g., [7,30,31] and others).

10. The available budget of investing in lost sales reduction is limited, i.e., $I(\alpha) \leq G$.

### 3. Model Formulation

In this section, we consider the continuous review inventory model with a mixture of backorders and lost sales, and assume that each lot contains a random number of defective items and controllable lead time. Upon order arrival, all goods are quickly inspected and all defective items in each lot are discovered and returned to the vendor at the time of delivery of the next lot. Wu and Ouyang [13] considered a $(Q, r, L)$ inventory model with defective items in an arrival lot and asserted the following function of the total expected cost per unit time, which is composed of ordering cost, non-defective holding cost, defective holding cost, stock-out cost, inspecting cost and lead time crashing cost. In mathematical symbolization, the problem is given by

$$
EAC(Q, r, L) = \frac{DA + C(L) + (\pi + \pi_\alpha E(X - r^*))}{Q[1 - E(p)]} + h[r - DL + \alpha E(X - r^*)] + h(Q - 1) \frac{E[p(1 - p)]}{1 - E(p)}
$$

\begin{equation}
+ \frac{h}{2} \left[ \frac{QV + Q\sigma r}{1 - E(p)} + \frac{E[p(1 - p)]}{1 - E(p)} \right] + \frac{Dv}{1 - E(p)}
\end{equation}

where $\alpha$ is the lost sales fraction, $p$ is the defective rate, and $X$ is the lead time demand which has a cumulative distribution function (c.d.f.) $F$ with finite mean $DL$ and standard deviation $\sigma \sqrt{L}$, where $\sigma^2$ denotes the variance of the demand per unit time.
In our new model, we consider the lost sales rate (or, equivalent to backorder rate) as a decision variable and try to minimize the sum of the capital investment cost of reducing lost sales rate and the inventory related costs by optimizing over \( Q, r, \alpha \) and \( L \) constrained \( 0 < \alpha \leq \alpha_0 \), where \( \alpha_0 \) is the original fraction of the shortage that will be lost. That is, according to our new model, the objective of our problem is to minimize the following total expected cost per unit time

\[
EAC(Q, r, \alpha, L) = il(\alpha) + EAC(Q, r, L)
\]  

(2)

here investment \( I(\alpha) \) is the one-time investment cost whose benefits will intend indefinitely into the future, thus the cost per unit time of such an investment is \( iI(\alpha) \), where \( i \) is fractional opportunity cost of capital per unit time. From assumption 10, the available budget is limited on \( I(\alpha) \leq G \), which implies \( \alpha_G \leq \alpha < \alpha_0 \), where \( \alpha_G = \alpha_0 \exp(-\delta G) \).

In this case the problem of determining the optimal ordering and investment strategies subject to a limited budget can be expressed as follows:

\[
\begin{align*}
\min_{(Q, r, \alpha, L)} & \quad EAC(Q, r, \alpha, L) = \frac{i}{\delta} \ln \left( \frac{\alpha_0}{\alpha} \right) \\
& + \frac{D[A + C(L) + (\pi + \pi_0(\alpha)E(X - r)^+)]}{Q[1 - E(p)]} \\
& + \frac{\sigma \sqrt{L}[k + \alpha \psi(k)] + h(Q - 1)\frac{E[p(1 - p)]}{1 - E(p)}}{2Q[1 - E(p)]}
\end{align*}
\]

subject to \( \alpha_G \leq \alpha < \alpha_0 \) 

(3)

We note that when the vendor promises that the arriving order contains no defective items and lead time is prescribed, our model (3) reduces to that of Ouyang and Chang [7].

### 3.1 Normal Distribution Case

In this subsection, we assume that the lead time demand \( X \) follows a normal distribution with mean \( DL \) and standard deviation \( \sigma \sqrt{L} \). From assumption 5, we can treat the safety factor \( k \) as a decision variable instead of the reorder point \( r \). Hence, the expected shortage quantity \( E(X - r)^+ \) at the end of the cycle can be expressed as a function of safety factor \( k \), i.e.,

\[
E(X - r)^+ = \int_{r}^{\infty} (x - r)dF(x) = \sigma \sqrt{L} \psi(k),
\]

(4)

where \( \psi(k) = \Phi(k) - k[1 - \Phi(k)] \)

where \( \Phi \) and \( \Phi \) denote the standard normal probability density function (p.d.f.) and cumulative distribution function (c.d.f.), respectively. This is done by direct calculations. Therefore, our problem can be transformed to

\[
\begin{align*}
\min_{(Q, k, \alpha, L)} & \quad EAC^N(Q, k, \alpha, L) = \frac{i}{\delta} \ln \left( \frac{\alpha_0}{\alpha} \right) \\
& + \frac{D[A + C(L) + \pi(\alpha)\sigma \sqrt{L} \psi(k)]}{Q[1 - E(p)]} \\
& + \frac{\sigma \sqrt{L}[k + \alpha \psi(k)] + h(Q - 1)\frac{E[p(1 - p)]}{1 - E(p)}}{2Q[1 - E(p)]}
\end{align*}
\]

subject to \( \alpha_G \leq \alpha < \alpha_0 \) 

(5)

where \( EAC^N(\cdot) \) is the total expected cost per unit time attained by normal distribution.

In order to solve this nonlinear programming problem, we first ignore the constraint \( \alpha_G \leq \alpha < \alpha_0 \) and take the first order partial derivatives of \( EAC^N(Q, k, \alpha, L) \) with respect to \( Q, k, \alpha \) and \( L \in (L_i, L_{i-1}) \), respectively. We obtain

\[
\begin{align*}
\frac{\partial EAC^N(Q, k, \alpha, L)}{\partial Q} &= -\frac{D[\pi(\alpha)\sigma \sqrt{L} \psi(k)]}{Q^2[1 - E(p)]} \\
+ \frac{\sigma \sqrt{L}[k + \alpha \psi(k)] + h(Q - 1)\frac{E[p(1 - p)]}{1 - E(p)}}{2Q[1 - E(p)]}
\end{align*}
\]

(6)

\[
\begin{align*}
\frac{\partial EAC^N(Q, k, \alpha, L)}{\partial k} &= \frac{D(\alpha)[\pi(\alpha)\sigma \sqrt{L} \psi(k)]}{Q[1 - E(p)]} \\
+ \frac{\sigma \sqrt{L}[k + \alpha \psi(k)]}{\alpha[Q[1 - E(p)]]}
\end{align*}
\]

(7)

\[
\begin{align*}
\frac{\partial EAC^N(Q, k, \alpha, L)}{\partial \alpha} &= -\frac{i}{\delta} \left( \frac{\sigma \sqrt{L} \psi(k)}{Q[1 - E(p)]} \right) \\
& + \frac{h(Q - 1)\frac{E[p(1 - p)]}{1 - E(p)}}{2Q[1 - E(p)]}
\end{align*}
\]

(8)

\[
\begin{align*}
\frac{\partial EAC^N(Q, k, \alpha, L)}{\partial L} &= \frac{D(\pi(\alpha)\sigma \sqrt{L} \psi(k))}{Q^2[1 - E(p)]} \\
+ \frac{\sigma \sqrt{L}[k + \alpha \psi(k)]}{2Q[1 - E(p)]}
\end{align*}
\]

(9)
By examining the second order sufficient conditions, it can be shown that $EAC^N(Q, k, \alpha, L)$ is not a convex function of $(Q, k, \alpha, L)$. However, for fixed $(Q, k, \alpha)$, $EAC^N(Q, k, \alpha, L)$ is concave in $L \in [L_0, L_{-1}]$, since

$$\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial L^2} = -\frac{\partial}{\partial L} \left( \frac{D(\pi + \alpha \sigma L^2 \psi(k))}{4Q[1 - E(p)]} \right) + \frac{1}{4} \frac{h\sigma L^2 [k + \alpha \psi(k)]}{Q[1 - E(p)]} < 0$$

Thus, the minimum total expected cost per unit time will occur at the end points of the interval $[L_0, L_{-1}]$. On the other hand, we note that for fixed $L$, the convexity of the total expected cost per unit time $EAC^N(Q, k, \alpha, L)$ is not guaranteed for the point $(Q, k, \alpha)$ by examining the second order sufficient conditions (see Ouyang and Chang [7]). However, for fixed $(\alpha, L)$, $EAC^N(Q, k, \alpha, L)$ is convex in $(Q, k)$ (see Appendix for detailed proof). Thus, fixed $L \in [L_0, L_{-1}]$ and $\alpha$, the minimum value of Eq. (5) will occur at the point that satisfies $\partial EAC^N(Q, k, \alpha, L)/\partial Q = 0$ and $\partial EAC^N(Q, k, \alpha, L)/\partial k = 0$, simultaneously. The resulting solutions are

$$Q = \left\{ \frac{2D[A + C(L) + (\pi + \alpha \sigma L^2 \psi(k))]}{hH} \right\}^{1/2} \tag{10}$$

where $H = 1 - 2E(p) + E(p^2) + \frac{2h'}{h} E[p(1 - p)].$

and

$$1 - \Phi(k) = \frac{h}{Q[1 - E(p)]} + \frac{\alpha L}{h} \tag{11}$$

For fixed $L \in [L_0, L_{-1}]$ and $\alpha$, we denote the optimal solution of $(Q, k)$ by $(Q^*_a, k^*_a)$. Next, we consider the constraint $\alpha_G \leq \alpha \leq \alpha_0$. Since $EAC^N(Q^*_a, k^*_a, \alpha, L)$ has a smooth curve for $\alpha \in [\alpha_G, \alpha_0]$, we can develop the following iterative algorithm to find the optimal $(Q^*, k^*, \alpha^*, L^*)$.

**Algorithm**

1. For each $L_i, i = 0, 1, 2, \ldots, m$, perform (i) to (ii).
2. For a given limited budget $G$ and original lost sales rate $\alpha_0$, we divide the interval $[\alpha_G, \alpha_0]$ into $n$ equal subintervals, where $\alpha_G = \alpha_0 \exp(-\delta G)$ and $n$ is large enough, and we let $\alpha_j = \alpha_0 - j(\alpha_0 - \alpha_G)/n, j = 0, 1, 2, \ldots, n$. (iii) Start with $k_i = 0$. (iv) Substituting $k_i$ into Eq. (10) to evaluate $Q_i$. (v) Utilizing $Q_i$ to determine $k_{i+1}$ from Eq. (11). (vi) Repeat (iv) and (v) until no change occurs in the values of $Q_i$ and $k_i$.

Denote the solution by $(Q^*_a, k^*_a, \alpha^*_a)$. Step 2. For each $(Q^*_a, k^*_a, \alpha^*_a, L_i), i = 0, 1, 2, \ldots, n$, calculate the corresponding total expected cost per unit time $EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$, utilizing Eq. (5).

3. Find $\min_{i = 0, 1, \ldots, n} EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$. If $EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i) = \min_{i = 0, 1, \ldots, n} EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$, then go to Step 4.

4. For each $(Q^*_a, k^*_a, \alpha^*_a, L_i), i = 0, 1, 2, \ldots, m$, calculate the corresponding total expected cost per unit time $EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$, utilizing Eq. (5).

5. Find $\min_{i = 0, 1, \ldots, m} EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$. If $EAC^N(Q^*_a, k^*_a, \alpha^*_a, L^*), L^*) = \min_{i = 0, 1, \ldots, m} EAC^N(Q^*_a, k^*_a, \alpha^*_a, L_i)$, then $(Q^*_a, k^*_a, \alpha^*_a, L^*)$ is the optimal solution and thus the optimal reorder point $r^* = DL^* + k^* \alpha^* L^*$ follows.

### 3.2 Distribution Free Model

The information about the probability distribution of lead time demand is often limited. Thus, in this subsection, we relax the assumption of a full knowledge about the lead time demand distribution and only assuming that the c.d.f. of the lead time demand $X$ belongs to the class $\mathbb{F}$ of the c.d.f.'s with a known finite mean $DL$ and standard deviation $\sigma \sqrt{L}$. Since the probability distribution of $X$ is unknown, we cannot find the exact value of the expected demand shortage quantity at the end of each cycle, $E(X - r)^*$. Therefore, we adopt the minmax distribution free procedure to find the least favorable c.d.f. $F$ in $\mathbb{F}$ for each $(Q, r, \alpha, L)$ and then minimize the total expected cost per unit time over $(Q, r, \alpha, L)$. Symbolically, our problem is to solve

$$\min_{(Q, r, \alpha, L)} \max_{F \in \mathbb{F}} EAC(Q, r, \alpha, L) \text{ Subject to } \alpha_j \leq \alpha \leq \alpha_0, \text{ where } \alpha_j = \alpha_0 \exp(-\delta G)$$

To this end, we need the following inequality; Gallego
and Moon [24] provided a tight upper bound to the total expected cost per unit of time.

For any $F \in F$

$$E[(X - r)^+] \leq \frac{1}{2} \left[ \sigma^2 L + (r - DL)^2 - (r - DL) \right]$$  \hspace{1cm} (13)

In other words, we can always find a distribution in which the above bound is satisfied with equality for every $r$. Let $k \geq 0$ be the safety factor as defined in the preceding subsection, our problem is to minimize the cost function for the worst distribution

$$EAC^W(Q, k, \alpha, L)$$  \hspace{1cm} (14)

where $EAC^W(\cdot)$ is the least upper bound of $EAC(\cdot)$.

Again, the analogous approach in the previous subsection is used to solve Eq. (14). We first ignore the constraint $a_0 \leq \alpha < a_0$, and find that, for fixed $(Q, k, \alpha)$, $EAC^W(Q, k, \alpha, L)$ is concave in $L\in[L_0, L_{-1}]$, since

$$\frac{\partial^2 EAC^W(Q, k, \alpha, L)}{\partial L^2} = \frac{D(\pi + \pi_\alpha \alpha) \sigma \sqrt{L} (\sqrt{1 + k^2} - k)}{8Q[1 - E(p)]}$$

where $\sigma = \frac{h}{\sqrt{1 + k^2}}$, and $\alpha \geq 0$, $a_0 < \alpha_0$.

Thus, for fixed $(Q, k, \alpha)$, the minimum total expected cost per unit time will occur at the end points of the interval $[L_0, L_{-1}]$. In addition, for fixed $(\alpha, L)$, it can be shown that $EAC^W(Q, k, \alpha, L)$ is convex in $(Q, k)$. Therefore, the minimum value of Eq. (14) will occur at the point which satisfies $\partial EAC^W(Q, k, \alpha, L)/\partial Q = 0$ and $\partial EAC^W(Q, k, \alpha, L)/\partial k = 0$ simultaneously. The resulting solutions are

$$Q = \left[ \frac{D(2[A + C(L)] + (\pi + \pi_\alpha \alpha) \sigma \sqrt{L} (\sqrt{1 + k^2} - k))}{hH} \right]^{\frac{1}{4}}$$  \hspace{1cm} (15)

and

$$1 - \frac{k}{\sqrt{1 + k^2}} = \frac{2hQ[1 - E(p)]}{D(\pi + \pi_\alpha \alpha) + h\alpha Q[1 - E(p)]}$$  \hspace{1cm} (16)

We denote the optimal point $(Q^{**}_o, k^{**}_o, \alpha^{**}_o)$.

4. Numerical Examples

Example 1

In order to illustrate the above solution procedure and the effects of investing in lost sales rate reduction, let us consider an inventory system with the data used in Ouyang and Chang [7]: $D = 2340$ units/year, $A = $250/ order, $h = $20/unit/year, $h' = $12/unit/year, $v = $1.6/unit, $\pi = $50/unit, $\pi_0 = $150/unit, $\sigma = 25$ units/week, where $1 \text{ year} = 52 \text{ weeks}$, $\delta = 0.1\%$, $i = 0.1 \text{ dollar/year}$, the lead time has three components with data shown in Table 1, and the defective rate $p$ in an order lot has a Beta distribution with parameters $s = 1$ and $t = 4$, i.e., the p.d.f. of $p$ is given by

$$g(p) = \begin{cases} 4(1 - p)^3, & 0 < p < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $E(p) = s/(s + t) = 0.2$, $E(p^2) = s(s + 1)(s + t + 1) = 0.0667$, and $\text{Var}(p) = E(p^2) - E^2(p) = 0.02667$.

Suppose that the lead time demand follows a normal dis-
We solve the case where budget $G = $2000 and let $\alpha_{jL} = \alpha_0 - j(\alpha_0 - \alpha_L)/n$, $j = 0, 1, 2, \ldots, n$, and take $n = 1000$. Applying the proposed Algorithm procedure yields the results shown in Table 2 for the lost sales ratio $\alpha_0 = 0.2, 0.4, 0.6, 0.8$. Furthermore, a summary of the optimal solutions are tabulated in Table 3 and we list the optimal results of not investing policy in the same table to illustrate the effects of investing in lost sales rate reduction. From the results in Table 3, comparing our new model with that of fixed lost sales rate case, we observe that savings, which range from 0.09% to 1.05% which implies that savings can be easily achieved due to controlling the lost sales rate through investment.

**Example 2.**

The data are the same as in Example 1, except that the probability distribution of the lead time demand is unknown. Applying the similar procedures as proposed in Algorithm, we obtain the results shown in Table 4, and the summarized optimal solutions are tabulated in Table 5. Also, the optimal results of not investing policy are listed in the same table for comparison. It follows from Table 5 that the savings range from 4.14% to 12.95% which implies that significant savings can be easily achieved due to controlling the lost sales rate through investment. Further, we compare the performance of distribution free approach against the normal distribution case. The optimal results of normal distribution and distribution free cases are $(Q^*, k^*, \alpha^*, L^*)$ and $(Q^{**}, k^{**}, \alpha^{**}, L^{**})$ respectively. Substituting them into Eq. (5), the cost of using $(Q^{**}, k^{**}, \alpha^{**}, L^{**})$ instead of the optimal $(Q^*, k^*, \alpha^*, L^*)$ for a normal distribution is $EAC^N(Q^{**}, k^{**}, \alpha^{**}, L^{**}) - EAC^N(Q^*, k^*, \alpha^*, L^*)$. This is the largest amount that one is willing to pay for the knowledge of the form of the lead time demand distribution. This quantity can be regarded as the expected value of additional

### Table 2. Solution procedures of Example 1 ($L_i$ in weeks)

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>$i$</th>
<th>$L_i$</th>
<th>$C(L_i)$</th>
<th>$\alpha'_0$</th>
<th>$I(\alpha'_0)$</th>
<th>$Q'_i$</th>
<th>$r'_i$</th>
<th>$EAC^N$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0.0723</td>
<td>1017.49</td>
<td>294</td>
<td>490</td>
<td>13462</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>5.6</td>
<td>0.0871</td>
<td>831.26</td>
<td>293</td>
<td>384</td>
<td>13094</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
<td>22.4</td>
<td>0.1144</td>
<td>558.62</td>
<td>297</td>
<td>274</td>
<td>12754*</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>57.4</td>
<td>0.1396</td>
<td>359.54</td>
<td>312</td>
<td>217</td>
<td>12786</td>
</tr>
<tr>
<td>0.4</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0.0721</td>
<td>1713.41</td>
<td>294</td>
<td>490</td>
<td>13532</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>5.6</td>
<td>0.0869</td>
<td>1526.71</td>
<td>293</td>
<td>384</td>
<td>13163</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
<td>22.4</td>
<td>0.1143</td>
<td>1252.64</td>
<td>297</td>
<td>274</td>
<td>12823*</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>57.4</td>
<td>0.1396</td>
<td>1052.68</td>
<td>312</td>
<td>217</td>
<td>12856</td>
</tr>
<tr>
<td>0.6</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0.0812</td>
<td>2000.00</td>
<td>294</td>
<td>490</td>
<td>13573</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>5.6</td>
<td>0.0864</td>
<td>1937.94</td>
<td>293</td>
<td>384</td>
<td>13204</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
<td>22.4</td>
<td>0.1144</td>
<td>1657.23</td>
<td>297</td>
<td>274</td>
<td>12863*</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>57.4</td>
<td>0.1398</td>
<td>1456.72</td>
<td>312</td>
<td>217</td>
<td>12896</td>
</tr>
<tr>
<td>0.8</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0.1083</td>
<td>2000.00</td>
<td>294</td>
<td>493</td>
<td>13608</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>5.6</td>
<td>0.1083</td>
<td>2000.00</td>
<td>293</td>
<td>385</td>
<td>13234</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
<td>22.4</td>
<td>0.1145</td>
<td>1944.04</td>
<td>297</td>
<td>274</td>
<td>12892*</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>57.4</td>
<td>0.1394</td>
<td>1747.26</td>
<td>312</td>
<td>217</td>
<td>12925</td>
</tr>
</tbody>
</table>

### Table 3. Summary of the optimal solutions of Example 1 ($L^*$ in weeks)

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>Lost sales rate reduction model</th>
<th>Fixed lost sales rate model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Budget limit $$2000$</td>
<td>No investment</td>
</tr>
<tr>
<td>$L^*$</td>
<td>$Q^*$</td>
<td>$r^*$</td>
</tr>
<tr>
<td>0.2</td>
<td>4</td>
<td>558.62</td>
</tr>
<tr>
<td>0.4</td>
<td>4</td>
<td>1252.64</td>
</tr>
<tr>
<td>0.6</td>
<td>4</td>
<td>1657.23</td>
</tr>
<tr>
<td>0.8</td>
<td>4</td>
<td>1944.04</td>
</tr>
</tbody>
</table>
information (EVAI) (Shore [32]). Moreover, the cost penalty is the ratio of the approximate expected cost to the optimal one and a summary is presented in Table 6. It can be observed from Table 6 that the cost performance of the distribution free approach is improving as \( \alpha_0 \) gets smaller and there exists a robust property of the EVAI as \( \alpha_0 \) varies.

5. Concluding Remarks

This paper considers a continuous review inventory model including controllable lead time and a random number of defective items in buyer’s arrival order lot with partial lost sales to effectively increase investment and to reduce the lost sales rate. Models with complete and partial information of lead time demand distribution are developed. By analyzing the total expected cost function, we develop an algorithm to determine the optimal order quantity, reorder point, the lost sales rate and lead time simultaneously. By framing the new models, we can obtain a significant amount of savings to increase the competitive edge in business. Two numerical examples

| Table 4. Solution procedures of Example 2 (\( L_i \) in weeks) |
| \( \alpha_0 \) | \( i \) | \( L_i \) | \( C(L_i) \) | \( \alpha_i \) | \( I(\alpha_i) \) | \( Q_i \) | \( r_i \) | \( EACW \) |
| 0.2 | 0 | 8 | 0 | 0.0271 | 2000 | 452 | 495 | 16916 |
| 1 | 6 | 5.6 | 0.0271 | 2000 | 430 | 390 | 16173 |
| 2 | 4 | 22.4 | 0.0271 | 2000 | 408 | 281 | 15339 |
| 3 | 3 | 57.4 | 0.0271 | 2000 | 405 | 223 | 15015* |
| 0.4 | 0 | 8 | 0 | 0.0541 | 2000 | 458 | 499 | 17144 |
| 1 | 6 | 5.6 | 0.0541 | 2000 | 436 | 394 | 16375 |
| 2 | 4 | 22.4 | 0.0541 | 2000 | 413 | 285 | 15508 |
| 3 | 3 | 57.4 | 0.0541 | 2000 | 409 | 226 | 15162* |
| 0.6 | 0 | 8 | 0 | 0.0812 | 2000 | 464 | 504 | 17363 |
| 1 | 6 | 5.6 | 0.0812 | 2000 | 441 | 398 | 16568 |
| 2 | 4 | 22.4 | 0.0812 | 2000 | 416 | 289 | 15670 |
| 3 | 3 | 57.4 | 0.0812 | 2000 | 412 | 230 | 15303* |
| 0.8 | 0 | 8 | 0 | 0.1083 | 2000 | 470 | 508 | 17572 |
| 1 | 6 | 5.6 | 0.1083 | 2000 | 445 | 403 | 16754 |
| 2 | 4 | 22.4 | 0.1083 | 2000 | 420 | 292 | 15826 |
| 3 | 3 | 57.4 | 0.1083 | 2000 | 416 | 233 | 15438* |

| Table 5. Summary of the optimal solutions of Example 2 (\( L^* \) in weeks) |
| \( \alpha_0 \) | \( L^* \) | \( \alpha^* \) | \( I(\alpha^*) \) | \( Q^* \) | \( r^* \) | \( EACW^W \) |
| 0.2 | 3 | 0.0271 | 2000 | 405 | 223 | 15015 |
| 0.4 | 3 | 0.0541 | 2000 | 409 | 226 | 15162 |
| 0.6 | 3 | 0.0812 | 2000 | 412 | 230 | 15303 |
| 0.8 | 3 | 0.1083 | 2000 | 416 | 233 | 15438 |

| Table 6. Calculation of EVAI |
| \( \alpha_0 \) | \( EAC^N(\alpha^*, \alpha^*, \alpha^*, L^*) \) | \( EAC^N(\alpha^*, \alpha^*, \alpha^*, L^*) \) | \( EVAI \) | \( \text{Cost penalty} \) |
| 0.2 | 13176 | 12754 | 422 | 1.033 |
| 0.4 | 13242 | 12823 | 419 | 1.033 |
| 0.6 | 13313 | 12863 | 450 | 1.035 |
| 0.8 | 13379 | 12892 | 487 | 1.038 |
are provided to demonstrate the results. The results in the numerical examples indicate that the significant savings can be easily achieved due to controlling the lost sales rate through investment. This research develops a more realistic inventory model, which can enhance the efficiency of an inventory manager in decision-making. For future research, it would be interesting to study the same inventory models under different general types of investment functions and their associated marginal cost behavior.

**Acknowledgements**

The author would like to thank the editor and two referees for carefully reading the paper and providing very helpful suggestions. This research was partially supported by NSC 100-2115-M-156-001 and 102-3114-C-156-001-ES.

**Appendix**

The proof of $EAC^N(Q, k, \alpha, L)$ is convex in $(Q, k)$ for fixed $\alpha$ and $L \in [L_i, L_{i-1}]$.

For fixed $\alpha$ and $L \in [L_i, L_{i-1}]$, we first obtain the Hessian matrix $H$ as follows.

$$H = \begin{bmatrix}
\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q^2} & \frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q \partial k} \\
\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial k \partial Q} & \frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial k^2}
\end{bmatrix}$$

Then we proceed by evaluating the principal minor determinant of $H$.

The first principal minor determinant of $H$ is $|H_{11}| > 0$, since

$$\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q^2} = \frac{2D}{Q^2} \left[ A + C(L) + (\pi + \pi_k)\sigma \sqrt{L} \psi(k) \right] > 0$$

(A.1)

and

$$\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial k^2} = \frac{D(\pi + \pi_k)\sigma \sqrt{L} \Phi(k)}{Q(1 - E(p))} > 0$$

(A.2)

Next, computing the second principal minor determinant of $H$, we obtain

$$\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q \partial k} = \frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q \partial k}$$

$$= - \frac{D(\pi + \pi_k)\sigma \sqrt{L} \Phi(k) - 1}{Q(1 - E(p))} > 0$$

(A.3)

and it follows from Equations (A.1)–(A.3) that

$$|H_{22}| = \begin{bmatrix}
\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q} & \frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial Q \partial k} \\
\frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial k \partial Q} & \frac{\partial^2 EAC^N(Q, k, \alpha, L)}{\partial k^2}
\end{bmatrix}$$

$$= \frac{2D}{Q^3} \left[ A + C(L) + (\pi + \pi_k)\sigma \sqrt{L} \psi(k) \right]$$

$$+ \frac{D(\pi + \pi_k)\sigma \sqrt{L} \Phi(k)}{Q(1 - E(p))} + \frac{D(\pi + \pi_k)\sigma \sqrt{L} \Phi(k)}{Q(1 - E(p))}$$

$$+ \frac{2D(\pi + \pi_k)^2 \sigma^2 \psi(k) \Phi(k)}{Q^3(1 - E(p))^2}$$

(A.4)

Let $G(k) = \psi(k) \psi(k) - [\Phi(k) - 1]^2$. Then $dG(k)/dk = -2k \psi(k) \psi(k) < 0$, since $\psi(k) > 0$ and $\psi(k) > 0$. Obviously, $\phi(\infty) = \lim \phi(k) = 0$, $\psi(\infty) = \lim \psi(k) = 0$ and $\Phi(\infty) = \lim \Phi(k) = 1$ so $G(\infty) = \lim G(k) = 0$, and also we have $G(k) > 0$, $\forall k \in [0, \infty)$ and so $|H_{22}| > 0$. Therefore, it is clearly seen that the Hessian matrix $H$ is positive definite at point $(Q, k)$.  

**Controlling Lost Sales Rate for Inventory System with Partial Backlogging Defective Items** 449
References


Controlling Lost Sales Rate for Inventory System with Partial Backlogging Defective Items


Manuscript Received: Nov. 21, 2011
Accepted: Oct. 12, 2013