Effects of Nano-particle Dampers on Multi-walled Carbon Nanotubes with Internal Resonance

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Abstract

This study simulates the vibrations of a single-walled CNT (SWCNT) and multi-walled CNTs (MWCNTs) resting on an elastic foundation using a hinged-hinged nonlinear multi-layered-beam with a tuned mass damper (as a nano-particle) installed on the beam. The beam is supported by cubic nonlinear springs to simulate the CNTs resting on an elastic matrix. We obtained the equations of motion for the nonlinear beam using Newton’s 2nd law with Euler’s angle transformation and Taylor series expansion. The frequency in each vibration mode of a SWCNT and MWCNTs resting on a nonlinear elastic matrix was analyzed by using the method of multiple scales (MOMS) to examine the possibility of internal resonance (I.R.). This study provides a novel concept of applying a nano-particle on the top layer of the CNTs to serve as a tuned mass damper (TMD). Our results indicate that internal resonance is possible in SWCNTs resting on an elastic matrix with a level of elasticity. However, van der Waals’ forces rule out the possibility of internal resonance in MWCNTs. Furthermore, vibration reduction can be achieved in both SWCNTs and MWCNTs through the placement of nano-particle in the correct location.

Key Words: Internal Resonance, Nonlinear Vibrations, Tuned Mass Damper, Carbon Nanotubes

1. Introduction

Since the recent emergence of nano-technology, researchers have been applying beams to simulate the vibration behavior of carbon nanotubes (CNTs). The most common application in composite materials involves the placement of CNTs on an elastic matrix. However, coupling between the elastic moduli of the foundation and the CNT can lead to internal resonance (I.R.). This phenomenon is difficult to detect and at small scales, deformation can be sufficiently large to damage the entire structure. This demonstrates the importance of research on this issue. Fu et al. [1] examined the nonlinear vibration of CNTs by adding the effects of stretching to a nonlinear Bernoulli-Euler beam in order to simulate the vibration behavior of a CNT resting on an elastic foundation. Shen [2] considered the vibration behavior of a nonlinear Bernoulli-Euler beam placed on a double-layered elastic foundation during post-buckling. That study demonstrated the profound effect that the stiffness of the elastic foundation can have on the vibrations of a nonlinear beam. Nayfeh and Pai [3] presented a 2D nonlinear Bernoulli-Euler beam model with stretching effects, suggesting that the effects of stretching should be included in the nonlinear hinged-hinged beam theory when dealing with CNT applications. To obtain an accurate prediction of mechanical performance, Pantano et al. [4] pointed out that the nonlinear geometric and material properties of nanostructures must be included in any investigating into their physical characteristics. Ru [5] investigated elastic buckling at the two ends of single-walled CNTs (SWCNTs) under extreme pressure. In many applications, CNTs are embedded in an elastic foundation or elastic matrix. Yoon et al. [6] investigated multi-walled CNTs (MWCNTs) resting on a Winkler type elastic foundation while taking into account the influence of van der Waals’...
forces. Khosrozadeh and Hajabasi [7] established a model for the analysis of nonlinear transverse free vibrations in an embedded double-walled CNT. Their model takes van der Waals’ forces into account and uses harmonic balance to examine the frequency response of CNTs. Hemmatnezhad and Aminikhah [8] employed a beam model similar to that used in Fu et al. [1] with an iterative approach to analyzing the frequency responses of nonlinear vibrations in single-walled, double-walled, and triple-walled CNTs. Simsek [9] analyzed the dynamic responses of embedded SWCNTs using non-local Timoshenko beam theory. A nano-particle was placed in the CNT to emulate dynamic loading. In vibrating objects, the added nano-particle can serve, to a certain degree, as a vibration absorber. Recent advances in technology indicate that controlling the location of a nano-particle on a CNT should be achievable within the foreseeable future. This motivated us to employ a nano-particle as a CNT vibration absorber.

In the analysis of vibrating bodies equipped with a dynamic vibration absorber (DVA), Wang and Chang [10] considered a hinged-free Bernoulli-Euler beam resting on an elastic foundation with nonlinear springs and linear damping. A time-dependent boundary dynamic vibration absorber (TDB-DVA) was hung at the free end of the beam, and the damping effects of the DVA on the beam were examined. Wang and Lu [11] proved that in a system with a hinged-hinged nonlinear beam resting on a nonlinear elastic foundation, 1:3 I.R. occurs within the 1st and 3rd modes when the ratio of the elastic modulus of the foundation to that of the beam is $\frac{9}{\pi^2}$. Wang and Kuo [12] considered vibrations in a hinged-free nonlinear beam placed on a nonlinear elastic foundation and discovered 1:3 I.R. in the 1st and 2nd modes in the system. Wang et al.[13] gave analytical solutions for a nonlinear tuned mass damper (TMD) and showed the damping effects most pronounced when the TMD located at the middle of the nonlinear beam. Wang and Liang [14] sought to identify the optimal damping effects of a lumped-mass vibration absorber (LMVA) applied to a hinged-hinged nonlinear beam resting on a nonlinear foundation. They identified the optimal parameter combination for the LMVA using a 3D maximum amplitude contour plot (3D MACP). Those researchers demonstrated that adjusting the position of the TMD or DVA and/or altering the mass, spring constant, and damping coefficient are feasible approaches to preventing I.R. and reducing vibration.

The present study derived the equations of motion using Newton’s 2nd law and nonlinear beam theory proposed by Nayfeh and Pai [3]. We obtained the equations of motion for a system comprising multi-layered nonlinear beams with van der Waals’ forces resting on an elastic foundation. We analyzed the nonlinear system using the method of multiple scales (MOMS) and determined the I.R. in the system using eigen analysis. A point mass-damper-spring TMD with a single degree of freedom (simulating the nano-particle) was then applied to the beam system to prevent systemic I.R. and reduce vibration. Instead of focusing on the tuning mechanism of nano-particle and the complexity of connecting to carbon nanotubes, the purpose of the present work is to provide a novel concept on vibration reduction of CNTs. Our results are presented using 3D graphs and the accuracy of our approach was verified using numerical methods.

2. Establishment of Theoretical Model

The present study considered a hinged-hinged nonlinear beam supported by nonlinear springs to simulate a system in which a CNT rests on an elastic foundation. Figure 1(a) presents a 3-D diagram of an MWCNT with a nano-particle resting on an elastic matrix; Figure 1(b) is the section view from Figure 1(a) and shows layer structure of the multi-walled-CNT. Figure 2 presents a diagram of the side cross-section of Figure 1(b). This vibration system includes multi-layered-beam with a TMD resting on an elastic foundation and the boundary conditions.

2.1 Deriving Nonlinear Equations of Motion

We referred to the nonlinear beam theory proposed by Nayfeh and Pai [3] using Newtonian’s 2nd law, Euler’s
angle transformation, and Taylor series expansion to develop a complete nonlinear beam model including the elastic foundation and the DVA. We excluded any rotation in the beam; i.e., limiting it to planar motions. According to “Nonlinear 2-D Euler-Bernoulli Beam Theory”, the equations of motion for the 2D beam are as follows:

\[ \rho \ddot{w} - EA \dddot{w} = EA \left( \frac{1}{2} \dddot{u}^2 - \dddot{w} \dddot{u} \right) \]
\[ + EI \left[ \dddot{w} \left( \dddot{u} - \dddot{w} \dddot{u} - 2 \dddot{w} \dddot{u} \dddot{w} - 3 \dddot{w} \dddot{w} \dddot{u} \right) \right] \]
\[ \rho \ddot{u} + E \dddot{w} + j_3 \ddot{u} = EA \left( \dddot{w} \dddot{u} - \dddot{u} \dddot{w} + \frac{1}{2} \dddot{u} \dddot{w} \right) \]
\[ + EI \left[ \dddot{w} \dddot{u} + (\dddot{w} \dddot{u}) \dddot{w} - (\dddot{u} \dddot{w}) \right] - \left( \dddot{w} \dddot{u} - \frac{1}{3} \dddot{u} \dddot{w} \right) \]  
\[ + F \]  
\[ (2) \]

where \( \rho \) is the density of the beam, \( E \) is the beam elastic coefficient, \( \dddot{u} \) and \( \dddot{w} \) are displacements of the beam along the \( y \)- and \( x \)-axes, respectively, and \( u \) and \( w \) are dimensionless displacements of \( \dddot{u} \) and \( \dddot{w} \) (please see Figure 2). \( A \) is the beam cross section area, \( I \) is the moment of inertia, \( (\cdot)' \) denotes the derivative with respective to space, \( (\cdot)^{'} \) represents the derivative with respective to time, and \( F \) is the uniform distributed load. The \( j_3 \) is the moment of inertia in the \( z \)-dir. (out-of-plane dir.). If the beam is not rotation, the \( j_3 \) can be neglected. The system examined in this study contains a slender elastic beam with an extremely small \( \rho / EA \). Thus, when Eq. (1) is divided by \( EA \), the longitudinal inertial force \( \rho \dddot{w} \) is reduced to a negligible level. The two ends of the beam are hinged and are not subject to external longitudinal forces. This necessitates taking into account the effects of stretching. We used the boundary conditions to determine the relationship between \( \dddot{u} \) and \( \dddot{w} \), wherein \( \dddot{w} \) is expressed as a function of \( \dddot{u} \). Thus, the equations of motion can be simplified into equations for \( \dddot{u} \). The boundary conditions of the beam are as follows:

\[ \dddot{w}(0, \dddot{t}) = 0, \quad \dddot{w}(\dddot{t}, \dddot{t}) = \dddot{P}(\dddot{t}), \quad \dddot{u} = 0, \quad \dddot{w} = 0 \]  
\[ (3) \]

\( P(\dddot{t}) \) represents the longitudinal external load on the point end of the beam and can be neglected in the present case of study. Based on Eq. (3), assuming that \( EI / EA \) and \( \rho / EA \) are extremely small, and neglecting the order of \( O(u^3) \), Eq. (1) can be rewritten as

\[ \dddot{u} = \left( -\frac{1}{2} \dddot{u}^2 \right) + \cdots \]  
\[ (4) \]

The integral of Eq. (4) is

\[ \dddot{w} = -\frac{1}{2} \dddot{u}^2 + a_1(\dddot{t}), \quad \dddot{w} = -\frac{1}{2} \int_0^\dddot{t} \dddot{u}^2 ds + a_1(\dddot{t})s + a_2(\dddot{t}) \]  
\[ (5) \]

Substituting Eq. (5) into the boundary conditions in Eq. (3) produces

\[ a_1(\dddot{t}) = 0, \quad a_1(\dddot{t}) = P(\dddot{t}) + \frac{1}{2 \dddot{t}} \int_0^{\dddot{t}} \dddot{u}^2 ds \]  
\[ (6) \]

Substituting Eqs. (5)–(6) into Eq. (2) makes it possible to simplify the equations of motion to an equation where vibrations are presented in the \( \dddot{u} \) direction. We included the structural damping term \( \mu \dddot{u} \) of the elastic beam and the van der Waals’ force \( (\dddot{e}_s (\dddot{u}_{s+1} - \dddot{u}_s) - \dddot{e}_s (\dddot{u}_s - \dddot{u}_{s-1})) \). We neglected \( P(\dddot{t}) \) and disregarded \( j_3 \dddot{w} \). The equation...
of motion was then rewritten as follows:

$$\rho A \ddot{u}_n + E \dddot{u}_n - \mu \dddot{u}_n = \frac{E A}{2} \int_0^{\pi} \dddot{u}_n^2$$

$$= \overline{c}_n (\dddot{u}_n - \dddot{u}_n)$$

Eq. (7) can be considered the general equation of motion for the middle layer of the CNT (denoted using subscript $n$, $n = 2 \sim N - 1$). To facilitate analysis, we non-dimensionalized the equations of motion. Eq. (7) of the middle layer (denoted using subscript $n$, $n = 2 \sim N - 1$), is divided by $EI A \overline{c}_1$. Let $u = \dddot{u} / \overline{c}_1$, $t = \tau \times \overline{c}_1$, $x = \overline{x} / \overline{c}_1$, and $\overline{c}_0 = (EI / \rho \overline{A} \overline{c}_1)^{1/2}$. Using the definitions of the nondimensionalized coefficients in Appendix 1, the dimensionless equation is as follows:

$$\dddot{u}_n + \dddot{u}_n^c + C \dddot{u}_n - \hat{A} \dddot{u}_n \left[ \int_0^1 (u_n')^2 \right]$$

$$= W_n [u_n - u_{n-1}] - W_{n-1} [u_n - u_{n-1}]$$

The top layer (denoted using subscript 1) and the bottom layer (denoted using subscript $N$) are discussed in the following.

### 2.1.1 Top Layer (denoted using subscript 1)

By using Newton’s 2nd law, the dimensionless equation of the TMD is

$$m_{MS} \ddot{u}_1 = \left[ \hat{k}_1 \left( u(x,t) - u_D \right) + \hat{g}_1 \left( \ddot{u}(x,t) - \ddot{u}_D \right) \right] = 0$$

where $m_{MS}$ is the ratio of the mass of the TMD and the elastic beam, $u_D$ is the dimensionless displacement of the mass of TMD, $l_D$ represents the location of the TMD, and $\hat{k}_1$ and $\hat{g}_1$ are the dimensionless spring constants and damping coefficients of the TMD and are defined as $\hat{k}_1 = k_1 / \rho \overline{I} \overline{c}_1^2$ and $\hat{g}_1 = g_1 / \rho \overline{I} \overline{c}_1^2$. Thus, the dimensionless equation of the top layer of the CNT is as follows:

$$\dddot{u}_n + \dddot{u}_n + C \dddot{u}_n - \hat{A} \dddot{u}_n \left[ \int_0^1 (u_n')^2 \right]$$

$$= W_n [u_n - u_{n-1}] - W_{n-1} [u_n - u_{n-1}]$$

### 2.1.2 Bottom Layer (denoted using subscript $N$)

Including the elastic foundation, the equation of the bottom layer of the CNT can be rewritten as

$$\dddot{u}_n + \dddot{u}_n + k u_n + \beta u_n + \hat{A} \dddot{u}_n \left[ \int_0^1 (u_n')^2 \right]$$

$$= -W_{n-1} [u_n - u_{n-1}]$$

The dimensionless boundary conditions are

$$u(0,t) = 0, \ u(l,t) = 0, \ u^c(0,t) = 0, \ u^c(l,t) = 0$$

### 2.2 Method of Multiple Scales (MOMS)

We adopted MOMS [3,11,13] to analyze the frequency response and determine the fixed points of the nonlinear equation. This approach involves dividing the time scale into fast and slow time scales. Suppose that $T_0 = \tau$ is the fast-time term, then $T_1 = \varepsilon^2 \tau$ is the slow-time terms and $u(x,t)$ can be expressed by time scales by letting $u(x,t) = u(x, t, \varepsilon)$. The $u(x,t, \varepsilon) = u_0(x, T_0, T_1, \ldots) + \varepsilon^3 u_1(x, T_0, T_1, \ldots)$, where $\varepsilon$ is the time scale of small disturbances and is a minimum value. We scale the dimensionless damping coefficient as $\varepsilon^2 C$, the coefficient of the stretching effect term as $\varepsilon^2 \hat{A}$, the nonlinear spring coefficient of the elastic foundation as $\varepsilon^2 \beta$, and the forcing term as $\varepsilon^3 F$. Disregarding the influence of high-order terms such as $\varepsilon^5, \varepsilon^6 \ldots$ on the system, we substitute these principles into Eq. (7) to obtain the expansion of the equation for each layer, as outlined in the following.

#### 2.2.1 Middle Layer (denoted using subscript $n$, $n = 2 \sim N - 1$)

The terms of the Order of $(\varepsilon^1)$ and $(\varepsilon^3)$ are

$$\varepsilon^1: \frac{\partial^2 u_{n0}}{\partial T_0^2} + u_{n0} + (W_{n0} + W_{n1}) u_{n0} - W_n u_{n0} + W_{n-1} u_{n-1} = 0$$

$$\varepsilon^3: \frac{\partial^2 u_{n1}}{\partial T_0^2} + u_{n1} + (W_{n0} + W_{n1}) u_{n1} - W_n u_{n1} + W_{n-1} u_{n-1} = 0$$

#### 2.2.2 Top Layer (denoted using subscript 1)

The terms of the Order of $(\varepsilon^1)$ and $(\varepsilon^3)$ are

$$\varepsilon^1: \frac{\partial^2 u_{10}}{\partial T_0^2} + u_{10} + W_{10} u_{10} - W_{10} u_{10} = 0$$

$$\varepsilon^3: \frac{\partial^2 u_{11}}{\partial T_0^2} + u_{11} + W_{10} u_{11} - W_{10} u_{11} = 0$$

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2.2.3 Bottom Layer (denoted using subscript \(N\))

The terms of the Order of \((/c101\) and \((/c101\)) are

\[
2.3 \text{ Analysis of System Modes}
\]

Using the separation of variables, we considered only the SWCNT and the elastic foundation when determining the mode shapes of the system. We first divided the transverse deformation \(u\) into time and space terms. Substituting \(u(x,t) = X(x)Y(t)\) into Eq. (17), and disregarding van der Waals’ forces, we can derive that

\[
XY/\delta_0/c38/c38 + XivY/\delta_0/c38/c38 + kXY = 0.
\]

From this we can obtain the natural vibration frequency of the system:

\[
\omega^2/c119/c97/c103/c103/c61/c43 = k/m,
\]

where \(\omega^2/c119/c97/c103/c103/c61/c43\) is an eigenvalue with \(\omega^2/c119/c97/c103/c103/c61/c43 = 1, 2, 3\ldots\). Using the boundary conditions, we can obtain the eigenfunction: \(\sin(\alpha_j x)\). Moreover, the mode shape can be expressed as \(\phi_j(x) = \sin(\alpha_j x)\).

3. Analysis of Internal Resonance Conditions

Before analyzing the influence of vibration on the CNTs with the TMD, we begin by examining basic nonlinear vibrations in the main body without TMD in order to characterize the nonlinear vibration behavior and our objective in damping.

3.1 I.R. Analysis of SWCNT Resting on an Elastic Foundation

In this section, we first discuss the I.R. conditions of an SWCNT resting on an elastic foundation. To derive the dynamic equation, we assume that

\[
\begin{align*}
\ddot{u}_{N1} &= \sum_{i=1}^{\infty} \phi_i(x) \xi_{N1i}(t),
\end{align*}
\]

After substituting the above into Eq. (18) and applying orthogonal properties, we multiple both sides by \(\phi_j\). Integrating from 0 to 1, we derive the following:

\[
\begin{align*}
\tilde{\xi}_{N1j} + \alpha_j \tilde{\xi}_{N1j} + k \tilde{\xi}_{N1j} + W_{N-1} \tilde{\xi}_{N1j} - \tilde{\xi}_{N1j} = 0,
\end{align*}
\]

where \(\tilde{\beta}_r = \int_0^1 \frac{\phi_j dx}{\int_0^1 \phi_j dx}, \tilde{A}_r = A \int_0^1 \phi_j^2 dx / \int_0^1 \phi_j^2 dx\), and \(\tilde{F}_r = f \int_0^1 \frac{\phi_j dx}{\int_0^1 \phi_j^2 dx}\).

As in the previous section, we can derive the natural frequency of the SWCNT resting on the elastic foundation: \(\omega_j = \sqrt{\alpha_j + k}\). The relationship between the natural frequency and the elastic modulus \(k\) of the elastic foundation is shown in Figure 3. I.R. is likely to occur when the frequency ratio is an integer. Figure 3 shows

**Figure 3.** Relationship between frequency ratio and \(k\).
that when \( k = 876 \), and \( \omega_1: \omega_2 = 1:3 \), then I.R. occurs. In the following, we seek to determine whether I.R. occurs in this circumstance. Based on Figure 3, we select modes 1 and 3 for I.R. Thus, we present the equations of motion of the 1st and 3rd modes as follows: For the 1st mode, let \( j = 1 \) in Eq. (20) and substitute the general solution for the generalized coordinate terms into the equation to obtain the following:

\[
\ddot{x}_{111} + \alpha_1 \dot{x}_{111} + k_x x_{111} + W_{111} \left( x_{111} - x_{111} \right) = -2\alpha_0 (B_{1x} e^{\omega_{1} t} e^{-\xi_1 t} + \omega_{1} x_{1x}) + \alpha_1 \left( B_{1x} e^{\omega_{1} t} e^{-\xi_1 t} + \omega_{1} x_{1x} \right) + (3B_{3x} e^{\omega_{3} t} e^{-\xi_3 t} + \omega_{3} x_{3x}) + \omega_{3} x_{3x} + (2B_{3x} e^{\omega_{3} t} e^{-\xi_3 t} + \omega_{3} x_{3x}) + \omega_{3} x_{3x}
\]

Similarly, let \( j = 3 \) in Eq. (20) and substitute the general solution for the generalized coordinate terms into the equation to obtain the 3rd mode equation and will not detail here.

### 3.1.1 Frequency Responses of System When \( k = 876 \)

According to Figure 3, there are many possibilities to trigger the I.R. when the frequency ratios of two modes equal the integer multiples. However, in the present work, the order of \( \varepsilon \) and \( \varepsilon' \) are chosen, there will be no I.R. in this system. The 1:3 I.R. becomes our main concern in this study. We examined the frequency ratio of 3 in Figure 3 and found that for the modes of 1st and 2nd, the modulus \( k \) to trigger I.R. is less than 200 and is too small for a suitable CNT matrix. Also, the \( k \) for the modes of 1st and 4th to trigger I.R. is too large (about 3000). Therefore, we took consideration of 1:3 I.R. of the 1st and 3rd modes for \( k = 876 \). When \( k = 876 \), than \( \omega_1: \omega_2 = 1:3 \), we obtain the secular terms of the 1st mode by factoring out the harmonic terms of \( \omega_1 \) and \( \omega_2 \) in Eq. (21). This makes it possible to obtain the secular terms of the 3rd mode by factoring out the harmonic terms of \( \omega_3 \) and \( \omega_4 \) in Eq. (21). Below, we separately discuss the excitation of the 1st and 3rd modes to obtain numerical solutions, draw fixed points plots, and determine whether I.R. appears. The frequency responses of excitation to 1st mode can be obtained as follows. Supposing that the frequency of the external force is \( \Omega = \omega_1 + \varepsilon \), then \( \dot{F}_1 \cdot e^{\varepsilon' t} = \dot{F}_1 \cdot e^{\omega_{1} t} e^{i\alpha t} = \dot{F}_1 \cdot e^{i\alpha t} e^{\omega_{1} t} \). Let \( \Gamma_1 = \sigma \dot{F}_1 + \zeta_1 \). In the 1st mode, we can derive the following:

\[
-2\omega_0 B_{N_1} - 2\omega_0 \dot{\zeta}_{11} B_{N_1} - (3\dot{B}_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx - Cia_0 B_{N_1} + (3A_0^2 B_{N_1} B_{N_1}) \int \phi_1^2 dx + (2A_0 B_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx + \left( 3\dot{B}_{N_1} B_{N_1} \dot{B}_{N_1} (\cos \Gamma_2 + i \sin \Gamma_2) \int \phi_1^2 dx \right)
\]

The sum of the square of the real and imaginary parts is

\[
\left[ -2\omega_0 \dot{\zeta}_{11} \dot{B}_{N_1} - (3\dot{B}_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx + (3A_0^2 B_{N_1} B_{N_1}) \int \phi_1^2 dx \right]^2 + (2A_0 B_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx + \cos \Gamma_2 (3\dot{B}_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx
\]

The real part of the 3rd mode is

\[
-2\omega_0 \dot{\zeta}_{11} \dot{B}_{N_1} - (3\dot{B}_{N_1} B_{N_1} \dot{B}_{N_1}) \int \phi_1^2 dx + (3A_0^2 B_{N_1} B_{N_1}) \int \phi_1^2 dx - \omega_1 x_{1x} + \omega_1 x_{1x}
\]

\[
\int \phi_1^2 dx - 6\dot{\beta}_1 \int \phi_1^2 dx = 0
\]
and the imaginary part is

\[ -2i \omega_1 \dot{\phi}_{13} - C i \omega_1 B_{23} - i \sin \Gamma_1 (3 \beta \overline{\beta}_1 B_{01}) \int_0^1 \phi_1 \dot{\phi}_1 \, dx = 0 \]  

(25)

Similarly, we can obtain the frequency responses of excitation to 3rd mode. This makes it possible to create fixed points plots using the solvability conditions. Figure 4 displays a fixed points plot resulting from excitation to the 1st mode, and Figure 5 shows a fixed points plot resulting from excitation to the 3rd mode. As shown in Figures 4 and 5, the vibration amplitudes in the 1st mode are greater than those in the 3rd mode regardless of whether excitation is applied to the 1st or 3rd mode. This is a typical nonlinear I.R. behavior, providing further evidence that I.R. indeed appears.

3.2 I.R. Analysis of MWCNT Resting on Elastic Foundation

The frequency of each layer in MWCNTs varies, and van der Waals’ forces must be taken into account when addressing the middle layer. Thus, the relation for frequencies must be derived from the equation of each layer. Once established, the \( \varepsilon^1 \) equation of each layer can be expressed using a matrix, such as the general formula below. For details on the parameters, please refer to Appendix 2. Using this matrix, we can obtain the system frequency to determine whether I.R. occurs. The general matrix for an MWCNT is as follows:

\[
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{22} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
\]

3.3 Frequency Responses of System When \( k = 876 \)

In section 3.1, internal resonance was found to occur in an SWCNT when \( k = 876 \). In this section, we discuss vibrations in a three-layered (\( N = 3 \)) MWCNT when \( k = 876 \), with the top layer being \( n = 1 \) subject to external force, and the bottom layer being \( n = 3 \) with the MWCNT resting on the elastic foundation. Based on Eqs. (13), (15), and (17), we can derive the \( \varepsilon^1 \) equation of each layer. Using the general matrix expression (Eq. (26)), we obtain the frequencies for each layer and mode, as follows: \( \omega_{11} = 9.889 \), \( \omega_{12} = 10.0013 \), \( \omega_{21} = 31.2155 \), \( \omega_{22} = 88.8286 \), \( \omega_{31} = 88.8412 \), and \( \omega_{33} = 93.633 \). We found no integer ratios among the frequencies, which means that theoretically I.R. does not appear in the MWCNT.

4. Analysis of System with TMD

This section extends on the results in sections 2 and 3 and analyzes the damping effects of a TMD applied to various locations on SWCNT and MWCNT resting on
an elastic foundation.

4.1 Frequency Analysis of SWCNT System with TMD

To verify the effects of adding a TMD to the system, we apply orthogonality on Eq. (16), selected the secular terms to obtain the solvability conditions. We use the frequency responses resulting from excitation to the 1st mode as an example. The frequency analysis of excitation to the 3rd mode is the same, so we will not repeat it. Suppose that the frequency of the external force is \( \Omega = \omega \pm \varepsilon \sigma \), then \( F_1 \cdot e^{i\omega t} = F_1 \cdot e^{i(\omega + \varepsilon \sigma) t} = F_1 \cdot e^{i\omega t} e^{i\varepsilon \sigma t} \). Let \( \Gamma = \sigma T_1 + \zeta_1 \) and \( \Gamma_2 = 3\zeta_1 - \zeta_2 \). For the 1st mode, the solvability condition of the CNT with the TMD produces the sum of the square of the real and imaginary parts is

\[
\sum_{n=1}^{3} (\text{re} \phi_n^2 + \text{im} \phi_n^2) = 0,
\]

The real part of the 3rd mode is the same as that of Eq. (24), and the imaginary part is the same as that of Eq. (25). The solvability conditions can then be used to draw fixed points plots (Figures 6 and 7) of the SWCNT with a TMD, which is discussed later.

4.2 Frequency Analysis of MWCNT System with TMD

I.R. does not appear in the MWCNT; therefore, we focus on the 1st and 3rd modes in the 1st layer with greater amplitudes when excitation is applied to the 1st mode in the 1st layer. Using the same steps used with SWCNT, we derive the solvability conditions of the MWCNT system with TMD. Supposing that the frequency of the external force is \( \Omega = \omega + \varepsilon \sigma \), then \( \omega \) is the natural frequency of the 1st mode in the 1st layer. Let \( F_1 \cdot e^{i\omega t} = F_1 \cdot e^{i\omega t} e^{i\varepsilon \sigma t} \), \( \Gamma = \sigma T_1 + \zeta_1 \), and \( \Gamma_1 = 0 \Rightarrow \zeta_1 = -\sigma \).

4.2.1 \( n = 1 \), Mode 1

Selecting the coefficients of the harmonic terms containing \( \omega_1 \) and multiplying them by \( e^{i\omega_1 t} \), we derive the sum of the square of the real and imaginary parts is

\[
\sum_{n=1}^{3} (\text{re} \phi_n^2 + \text{im} \phi_n^2) = 0.
\]

4.2.2 \( n = 1 \), Mode 3

The real part is

\[
\sum_{n=1}^{3} (\text{re} \phi_n^2 + \text{im} \phi_n^2) = 0.
\]
and the imaginary part is

\[-2o_{13} \dot{\psi}_{13} B_{13} \cos(-\psi_{13}) + C o_{13} B_{13} \sin(-\psi_{13})
+ \int_0^\infty \phi_{13}^2 dx (2 \dot{A}_{13} B_{13} \ddot{B}_{13} B_{13} \cos(-\psi_{13}))
+ \int_0^\infty \phi_{13}^2 dx (3 \dot{A}_{13} B_{13} \ddot{B}_{13} B_{13} \cos(-\psi_{13}))
+ \left\{ \frac{m_{13}^2 \dot{w}_{13} \omega_{13}^3 + \dot{g}_{13} \omega_{13} g_{13} B_{13}}{m_{13}^2 \omega_{13}^4 - 2m_{13} \omega_{13} \dot{w}_{13} \dot{g}_{13} + \dot{g}_{13}^2 \omega_{13}^2 + k_{13}^2} \right\} \phi_{13}
+ \dot{g}_{13} \left\{ \frac{B_{13} \dot{w}_{13} \omega_{13} g_{13} B_{13}}{m_{13}^2 \omega_{13}^4 - 2m_{13} \omega_{13} \dot{w}_{13} \dot{g}_{13} + \dot{g}_{13}^2 \omega_{13}^2 + k_{13}^2} \right\} \phi_{13}
\delta(x - \ell_{13}) \sin(-\psi_{13})
\right]
(29)
and the imaginary part is

\[-2o_{13} \dot{\psi}_{13} B_{13} \cos(-\psi_{13}) + C o_{13} B_{13} \sin(-\psi_{13})
+ \int_0^\infty \phi_{13}^2 dx (2 \dot{A}_{13} B_{13} \ddot{B}_{13} B_{13} \cos(-\psi_{13}))
+ \int_0^\infty \phi_{13}^2 dx (3 \dot{A}_{13} B_{13} \ddot{B}_{13} B_{13} \cos(-\psi_{13}))
+ \left\{ \frac{m_{13}^2 \dot{w}_{13} \omega_{13}^3 + \dot{g}_{13} \omega_{13} g_{13} B_{13}}{m_{13}^2 \omega_{13}^4 - 2m_{13} \omega_{13} \dot{w}_{13} \dot{g}_{13} + \dot{g}_{13}^2 \omega_{13}^2 + k_{13}^2} \right\} \phi_{13}
+ \dot{g}_{13} \left\{ \frac{B_{13} \dot{w}_{13} \omega_{13} g_{13} B_{13}}{m_{13}^2 \omega_{13}^4 - 2m_{13} \omega_{13} \dot{w}_{13} \dot{g}_{13} + \dot{g}_{13}^2 \omega_{13}^2 + k_{13}^2} \right\} \phi_{13}
\delta(x - \ell_{13}) \cos(-\psi_{13})
\right] \]

The equations for the 2nd and the 3rd layers are the same as the case of no TMD. Then, based on the solvability conditions, we can draw the fixed points plots of the MWCNT with a TMD, as discussed later.

5. Results and Discussion

5.1 Damping Effects on SWCNT
Figure 6 and 7 present fixed points plots of the 1st and 3rd modes in the SWCNT system equipped with a TMD ($m_{MS} = 0.05, \ell_D = 0.2, \dot{g}_s = 0.9, \dot{k}_s = 9$). Comparisons of Figure 4 with Figure 6 and Figure 5 with Figure 7 show that when excitation is applied to the 1st mode, the vibration amplitudes in the 1st and 3rd modes are smaller than those in the system without a TMD. When excitation is applied to the 3rd mode, the vibration amplitudes are greater than those in the 1st mode, and the amplitudes in both modes are smaller than those in the system without a TMD. This demonstrates that the TMD can indeed reduce system vibrations and prevent I.R. when the 3rd mode is excited. Thus, this section discusses amplitudes in the 1st mode when excitation is applied to the 1st mode and analyzes the amplitudes in the CNT resulting from various TMD locations, masses, spring constants, and damping coefficients to identify the optimal combination of parameters. Using the results in section 4, we compiled the fixed points plots of various parameter combinations ($m_{MS} = 0.001, 0.005, 0.01, 0.05$, and $0.1, \ell_D = 0.1-0.5, \dot{g}_s = 0.1-0.9, \dot{k}_s = 1-9$) and extracted the maximum amplitudes from the frequency response graphs. Due to the difficulties associated with this data of observation, we compiled 3D amplitude graphs using various $\dot{g}_s$ values and $\dot{k}_s = 1, 5, and 9$, as shown in Figure 8. The $x, y, and z$ axes indicate the mass ratio, TMD location, and corresponding maximum amplitude, respectively. The graphs use different colors to differentiate the amplitudes; red indicates greater amplitudes and blue signifies smaller amplitudes associated with better damping effects.

Take Figure 8 as an example, the 3D graph of maximum amplitudes resulting from $\dot{g}_s = 0.1$ and $\dot{k}_s = 1-9$. For the mesh surface outlined in red, $\dot{k}_s = 1$, whereas $\dot{k}_s = 5 and 9$ for the mesh surfaces outlined in blue and black, respectively. The brown mesh surface presents the am-
amplitudes in the SWCNT not equipped with a TMD. Figure 8(a) shows that regardless of the $\tilde{k}_s$ value, the minimum amplitudes appear with TMD locations at 0.2 and 0.5 and near $m_{MC} = 0.05$. Figure 8(b) presents the same trend. Furthermore, as $\tilde{g}_s$ increases (Figure 8(b), $\tilde{g}_s = 0.9$), the damping effects improve. Thus, we can conclude that a TMD with greater $\tilde{g}_s$ and $\tilde{k}_s$ values is better able to absorb vibrations in the main body.

To further analyze the damping effects of TMD parameter combinations, we compiled a 3D maximum amplitude contour plot (3D MACP) for the following combinations: $\tilde{g}_s = 0.1$ with $\tilde{k}_s = 1, 9$ and $\tilde{g}_s = 0.9$ with $\tilde{k}_s = 1, 9$ (Figures 9 and 10). Figures 9 and 10 are maximum amplitude contour plots for SWCNTs with different combinations of TMD positions and mass ratios. The blue area represents the CNTs lowest amplitude which has the best TMD damping effect. The red area has the highest amplitude. The vertical and the horizontal axes of the

![3D amplitude plot of SWCNT, excitation to the 1st mode.](image)

Figure 8. 3D amplitude plot of SWCNT, excitation to the 1st mode.

![3DMACP of SWCNT, excitation to the 1st mode, $\tilde{g}_s = 0.1$.](image)

Figure 9. 3DMACP of SWCNT, excitation to the 1st mode, $\tilde{g}_s = 0.1$. 
3DMACPs represent the TMD position and the mass ratio, respectively. The effects of TMDs mass ratios and positions for different $g_s$ and $k_s$, can be seen by comparing those 3DMACPs (Figures 9 and 10). By investigating the blue area, we can find the best damping parameter combinations for the TMD. As shown in both figures, the amplitude intervals for $k_s = 9$ are smaller than those for $k_s = 1$. A comparison of Figures 9(b) and 10(b) also shows that when $k_s = 9$, the amplitude intervals for $g_s = 0.9$ are smaller than those for $g_s = 0.1$. Moreover, the optimal damping effects all appear at $\ell_D = 0.2$ and $0.5$ with $m_{MS} = 0.05$. Extremely small or large $m_{MS}$ values do not produce good damping effects, nor when the TMD is close to the end of the CNT ($\ell_D = 0.001$). With an extremely small $m_{MS}$ and TMD close to the end of the CNT, these circumstances are similar to those without a TMD, and therefore have poor damping effects. In contrast, an overly large $m_{MS}$ ($m_{MS} = 0.1$) means an additional load on the CNT, which diminishes the effect of the TMD. At present, the optimal mass for the TMD is $m_{MS} = 0.05$. One interesting point is that $\ell_D = 0.2$ and $0.5$ produce optimal damping effects, regardless of the $g_s$ and $k_s$ values. This is because the CNT combines the 1st and 3rd modes, and from the perspective of beam vibration, $\ell_D = 0.2$ and $0.5$ are precisely where the maximum amplitudes of the combined mode shapes of the 1st and 3rd modes are situated, as shown in Figure 11. Thus, the best damping effects are obtained by placing the TMD where the maximum amplitudes are located. This result supports the findings of Wang and Kuo [12].

5.2 Damping Effects on MWCNT

The damping analysis of MWCNTs is identical to that of SWCNTs, as outlined in the following. Section
4.2 demonstrates that the TMD can indeed reduce the amplitude of vibrations within the system. We next discuss the 1st and 3rd modes in the 1st layer, which has greater amplitudes, and examine the damping effects of various parameter combinations to identify the optimal combination. Using the same analysis method as before, we compiled fixed points plots of various parameter combinations ($m_{\text{MS}} = 0.001, 0.005, 0.01, 0.05, \text{and } 0.1$, $\ell_D = 0.1~0.5$, $\tilde{D} = 0.1~0.9$, $\tilde{k}_s = 0.1~0.9$, $\tilde{k}_s = 1~9$) and extracted the maximum amplitudes from each frequency response graph.

We compiled 3D amplitude graphs of the 1st mode in the 1st layer and the 3rd mode in the 1st layer using various $\tilde{g}_s$ values and $\tilde{k}_s = 1, 5, \text{and } 9$.

To further analyze the damping effects of various TMD parameter combinations, we compiled 3D MACPs of the 1st and 3rd modes in the 1st layer for the combinations $\tilde{g}_s = 0.1$ with $\tilde{k}_s = 1, 9$ and $\tilde{g}_s = 0.9$ with $\tilde{k}_s = 1, 9$. Figures 12 and 13 present 3D MACPs of the 1st mode in the 1st layer. As can be seen in Figures 12 and 13, the amplitude intervals for $\tilde{k}_s = 9$ are smaller than those for $\tilde{k}_s = 1$.
A comparison of Figures 12(b) and 13(b) also show that when $\tilde{k}_s = 9$, the amplitude intervals for $\tilde{g}_s = 0.9$ are smaller than those for $\tilde{g}_s = 0.1$. Moreover, all of the optimal damping effects appear at $\ell_D = 0.2$ and 0.5 with $m_{MS} = 0.05$. Extremely small or large $m_{MS}$ values do not produce good damping effects, nor can this be achieved when the TMD is close to the end of the CNT ($\ell_D = 0.001$). At present, the optimal mass for the TMD is still $m_{MS} = 0.05$. The 3D MACPs of the 3rd mode in the 1st layer present the same trend which will not detail here. The damping effects of TMDs on the MWCNT and SWCNT are the same. We revealed that the same combinations of the TMD have optimal damping effects on both MWCNT and SWCNT. We also found that the amplitudes in the 1st layer of the MWCNT are always greater than those in the SWCNT. This is because the SWCNT is in direct contact with the elastic foundation, which is able to absorb vibrational energy from the SWCNT, thereby reducing amplitudes to smaller than those in the MWCNT.

To verify the accuracy of the 3D MACPs, we conducted numerical analysis of Figure 13(b), in which is shown the best damping effects for the 1st mode in the 1st layer. Orthogonalizing the equations of motion for the MWCNT produces the following equations.

\[
\begin{align*}
\ddot{\xi}_{11} + \alpha_1 \dot{\xi}_{11} + C_{\xi_{11}} + \int_0^1 \phi_1^2 d\xi_{11} + \int_0^1 \phi_1^2 d\xi_{11} + W_1(\xi_{11} - \tilde{\xi}) & + \tilde{k}_s(\tilde{\phi}_{11} + \dot{\phi}_{11}) + \tilde{g}_s(\tilde{\phi}_{11} + \tilde{\phi}_{11}) - \dot{\mu}_{11}) \\
\ddot{\xi}_{21} + \alpha_1 \dot{\xi}_{21} + C_{\xi_{21}} + \int_0^1 \phi_2^2 d\xi_{21} + \int_0^1 \phi_2^2 d\xi_{21} + \int_0^1 \phi_2^2 d\xi_{21} + W_2(\xi_{21} - \tilde{\xi}) & + \tilde{k}_s(\tilde{\phi}_{21} + \dot{\phi}_{21}) + \tilde{g}_s(\tilde{\phi}_{21} + \tilde{\phi}_{21}) - \dot{\mu}_{21}) & = 0
\end{align*}
\]

Using Eqs. (31)–(33), the equation of motion of TMD Eq. (9), and the Runge-Kutta (RK4) approach, we generated a time response plot of the system and determined whether it was consistent with the fixed points plots. Figure 14 displays the fixed points plots and time response plots of the 1st mode in the 1st layer resulting from $m_{MS} = 0.05$, $\ell_D = 0.2$, $\tilde{g}_s = 0.9$, and $\tilde{k}_s = 9$. As can be seen, the amplitudes in the time response plot are identical to those in the fixed points plots, which verifies the high accuracy of the proposed 3D MACP.

6. Conclusion

This study simulated the vibrations of SWCNTs and MWCNTs resting on an elastic foundation using a hinged-hinged nonlinear beam with a TMD (as a nano-particle) installed on the beam. We altered the mass, location, and $\tilde{g}_s$, $\tilde{k}_s$ values of the TMD and investigated the effects on the vibrations of the SWCNT and MWCNT systems. The results were analyzed using MOMS, fixed point plots, and 3D MACPs, the accuracy of which was verified using the Runge-Kutta (RK4) method. Finally, we arrived at the following conclusions based on the results of this study:

1. When the elastic modulus of the elastic foundation is $k = 876$, 1:3 I.R. appears in the 1st and 3rd modes in the SWCNT but can be avoided using a TMD.
2. Due to van der Waals’ forces between layers, I.R. does not occur in MWCNTs resting on an elastic foundation; however, applying a TMD can still reduce system vibrations.
3. With either a SWCNT or a MWCNT, the optimal damping effects appear when a TMD with $m_{MS} = 0.05$, $\ell_D = 0.2$ and 0.5, $\tilde{g}_s = 0.9$, and $\tilde{k}_s = 9$ is applied.
4. The location of the TMD is crucial to the effectiveness of system damping. From the perspective of beam vibration, $\ell_D = 0.2$ and 0.5 are precisely where the maximum amplitudes of the combined mode shapes of the 1st and 3rd modes are situated. Thus, placing the TMD in the locations associated with the maximum amplitudes provides the best damping effects.

![Figure 14. Verification of the accuracy of the MWCNT 3D MACP of Figure 13(b).](image-url)
5. TMD damping in the 1st layer of the MWCNT is not as effective as that on the SWCNT because SWCNT is in direct contact with the foundation, which can absorb vibrations.

Appendix

Appendix 1. Definitions of the Nondimensionalized Coefficients

\[ u = \frac{\bar{u}}{\ell}, \quad t = \frac{T}{\ell}, \quad x = \frac{x}{\ell}, \quad \bar{\alpha} = \left( \frac{EI}{\rho A \ell^4} \right)^{\frac{1}{2}}, \quad C = \frac{\bar{u}^2}{E I \left( \frac{EI}{\rho A} \right)^{\frac{1}{2}}}, \]

\[ \ddot{A} = \frac{A \ddot{\ell}}{2I}, \quad W_n = \frac{c \ell^4}{EI}, \quad k = \frac{k \ell^4}{EI}, \quad \beta = \frac{k \beta \ell^6}{EI}. \]

Appendix 2. Definitions of the Elements in the MWCNT Matrix

\[ M_{11} = \int_0^1 \dot{\phi}_1^2 dx, \quad M_{22} = \int_0^1 \dot{\phi}_2^2 dx, \quad M_{(2n-1)(2n-1)} = \int \phi_{2n-1}^2 dx, \]

\[ M_{(2n)(2n-1)} = \int \phi_{2n}^2 dx, \quad M_{(2n-1)(2n-1)} = \int \phi_{2n-1}^2 dx, \quad M_{(2n)(2n)} = \int \phi_{2n}^2 dx, \]

\[ K_{12} = \int \dot{\phi}_1 \dot{\phi}_2 dx, \quad K_{13} = -W_n \int \dot{\phi}_2 dx, \quad K_{21} = \int \phi_2 \phi_1 dx, \]

\[ K_{22} = 0, \quad K_{23} = \int \phi_2 \phi_{2n} dx + \int \phi_{2n} dx \ddot{\phi}_2 dx - W_n \int \dot{\phi}_2 dx, \]

\[ K_{2n} = -W_n \int \dot{\phi}_n dx, \quad K_{(2n-1)(2n-1)} = -W_n \int \dot{\phi}_{2n-1} dx, \quad K_{(2n)(2n)} = -W_n \int \dot{\phi}_{2n} dx, \]

\[ K_{(2n-1)(2n-2)} = 0, \quad K_{(2n)(2n-1)} = 0, \quad K_{(2n)(2n-2)} = -W_{n-1} \int \dot{\phi}_{2n-1} dx, \]

\[ K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx + (W_n + W_{n-1}) \int \phi_{2n} dx, \]

\[ K_{(2n-1)(2n-2)} = 0, \quad K_{(2n)(2n-1)} = 0, \quad K_{(2n)(2n-2)} = -W_{n-1} \int \dot{\phi}_{2n-2} dx, \]

\[ K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx + (W_n + W_{n-1}) \int \phi_{2n} dx, \]

\[ K_{(2n-1)(2n-2)} = 0, \quad K_{(2n)(2n-1)} = 0, \quad K_{(2n)(2n-2)} = W_{n-1} \int \dot{\phi}_{2n-2} dx, \]

\[ K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n)} = \int \phi_2 \phi_{2n} dx + (W_n + W_{n-1}) \int \phi_{2n} dx, \]

\[ K_{(2n-1)(2n-3)} = -W_{n-1} \int \dot{\phi}_{2n-3} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

\[ K_{(2n)(2n-2)} = \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = 0, \quad K_{(2n)(2n-2)} = 0, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n)(2n-2)} = \frac{1}{0} \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = \frac{1}{0} \int \phi_2 \phi_{2n-1} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

\[ K_{(2n)(2n-2)} = \frac{1}{0} \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = \frac{1}{0} \int \phi_2 \phi_{2n-1} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

\[ K_{(2n)(2n-2)} = \frac{1}{0} \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = \frac{1}{0} \int \phi_2 \phi_{2n-1} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

\[ K_{(2n)(2n-2)} = \frac{1}{0} \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = \frac{1}{0} \int \phi_2 \phi_{2n-1} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

\[ K_{(2n)(2n-2)} = \frac{1}{0} \int \phi_2 \phi_{2n} dx, \quad K_{(2n)(2n-3)} = 0, \]

\[ K_{(2n-1)(2n-3)} = \frac{1}{0} \int \phi_2 \phi_{2n-1} dx, \quad K_{(2n-1)(2n-2)} = 0, \]

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