Risk Evaluation of Product Considering Measurement Error

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Abstract

The measurement error has the important effect on the decisions made in process capability studies, process setting, process control and inspection. In this paper, we further consider the problem that the true value and the observed value of product characteristic are dependent under the measurement error. By solving the total risk function of product, we can obtain the maximum total risk under the given specification limits of customers.

Key Words: Measurement Error, Specification Limits, Bivariate Normal Distribution

1. Introduction

The true value of product characteristic is not equal to its observed value when the measurement error exists. The operator error and the gauge error are two components of measurement error. Both these components affect both the accuracy and precision of the measurement system. Because of the measurement error, the decision to accept or reject a product is made based on the location of the observed value, \( Y \), relative to the acceptance and rejection regions and is not based on the location of the true value, \( X \). As a result, there are two types of possible wrong decisions: (1) type I error—reject a good unit and (2) type II error—accept a defective unit.

Basnet and Case [1] and Chandra [2] addressed that the decision problems of the true value of product and the measurement error are independent of each other. However, it might be true in most real-life applications that the true value and the observed value of product characteristics are dependent under the measurement error. In this paper, we further present the modified Chandra’s [2] independent model. By solving the total risk function of product, we can obtain the maximum total risk under the given specification limits of customers.

2. Chandra’s Model

Let the true value of the quality characteristic measured be \( X \), the observed value of the characteristic be \( Y \), and the measurement error be \( M \). The relationship among \( X, Y, M \) is \( Y = X + M \). The random variables \( X \) and \( V \) are independent of each other and they are assumed to be normally distributed, respectively. Assume that \( X \) is a nominal-the-best type characteristic with the lower specification limit (LSL) and the upper specification limit (USL).

From Chandra [2], we have the probability of rejecting a good unit is

\[
\alpha = P(Y < \text{LSL} \text{ or } Y > \text{USL}, \text{ when } \text{LSL} \leq X \leq \text{USL})
\]

\[
= \int_{\text{LSL}}^{\text{USL}} f(x)dx \left[ \int_{\text{LSL}}^{\text{USL}} \Phi \left( \frac{\text{USL} - x}{\sigma_M} \right) - \Phi \left( \frac{\text{LSL} - x}{\sigma_M} \right) \right].
\]

(1)

where \( x \) is the given true value; \( \sigma_M \) is the standard deviation of measurement error; \( f(x) \) is the probability density function of normal random variable \( X \); \( \Phi(\cdot) \) is the cumulative distribution function of standard normal random variable.

From Chandra [2], we have the probability of accepting a defective unit is

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\[ \beta = P(LSL \leq Y \leq USL, \text{ when } X < LSL \text{ or } X > USL) \]
\[ = \int_{-\infty}^{LSL} \left[ \Phi \left( \frac{USL - X}{\sigma_M} \right) - \Phi \left( \frac{LSL - X}{\sigma_M} \right) \right] f(x) dx + \int_{USL}^{\infty} \left[ \Phi \left( \frac{USL - X}{\sigma_M} \right) - \Phi \left( \frac{LSL - X}{\sigma_M} \right) \right] f(x) dx \]

(2)

where \( x, \sigma_M, f(x) \), and \( \Phi(\cdot) \) are the same as Eq. (1).

Chandra [2] pointed out that as \( \sigma_M \) decreases (as the precision of the measurement system improves), the above \( \alpha \) and \( \beta \) will approach zero, respectively. Because of the measurement error including operator error and gauge error, one needs to reduce the variance of operator reproducibility and gauge repeatability in order to reduce the variance of measurement error.

3. Modified Chandra’s Model

Let \((X, Y)\) have a standard bivariate normal distribution with mean \(\mu_X = \mu_Y = 0\), variance \(\sigma_X^2 = \sigma_Y^2 = 1\), and a correlation coefficient \(\rho\). Since the conditional distribution of \(Y\) given \(X = x\) is also a normal distribution with mean \(px\) and variance \(1 - \rho^2\), the normal regression equation is given by \(Y = \rho X + M\), where \(M\) is the measurement error term with a normal distribution with mean zero and variance \(1 - \rho^2\).

The probability of rejecting a good unit is the probability that the observed value, \(Y\), falls in the rejection region when the true value, \(X\), is in the acceptance region. Hence, the probability of type I error, \(\alpha\), is

\[ \alpha = P(Y < LSL \text{ or } Y > USL, \text{ when } LSL \leq X \leq USL) \]
\[ = \int_{LSL}^{USL} \left[ \int LSL \Phi(y | x) dy + \int \Phi(y | x) dy \right] \phi(x) dx \]
\[ = \int_{LSL}^{USL} \left[ \int d\Phi \left( \frac{y - px}{\sqrt{1 - \rho^2}} \right) + \int d\Phi \left( \frac{y - px}{\sqrt{1 - \rho^2}} \right) \right] \phi(x) dx \]
\[ = \int \left[ \Phi \left( \frac{LSL - px}{\sqrt{1 - \rho^2}} \right) + 1 - \Phi \left( \frac{USL - px}{\sqrt{1 - \rho^2}} \right) \right] \phi(x) dx \]
\[ = \int \phi(x) dx - \left[ \int \Phi \left( \frac{USL - px}{\sqrt{1 - \rho^2}} \right) - \Phi \left( \frac{LSL - px}{\sqrt{1 - \rho^2}} \right) \right] \phi(x) dx \]

\[ \phi(x) dx \]

(3)

where \( LSL \) is the lower specification limit; \( USL \) is the upper specification limit; \( X \) is the true value of product characteristic; \( Y \) is the observed value of product characteristic; \( \phi(\cdot) \) is the probability density function of standard normal random variable \(X\); \( \Phi(\cdot) \) is the cumulative distribution function of standard normal random variable \(X\); \( \phi \left( y \mid x \right) \) is the probability density function of standard normal conditional distribution of \(Y\) given \(X = x\).

Let \( BVN (p, q, \rho) \) denote the standard bivariate normal cumulative distribution function. We have

\[ BVN (p, q, \rho) \]
\[ = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right) dx dy \]

(4)

From Owen [3], we obtain the following three equations:

1. \[ \int_{-\infty}^{\infty} \Phi(a + bx) \phi(x) dx = BVN \left( \frac{a}{\sqrt{1 + b^2}}, k, \frac{-b}{\sqrt{1 + b^2}} \right) \]

(5)

2. \[ \int_{-\infty}^{\infty} \Phi(a + bx) \phi(x) dx = BVN \left( \frac{-a}{\sqrt{1 + b^2}}, -k, \frac{b}{\sqrt{1 + b^2}} \right) \]

(6)

3. \[ \int_{-\infty}^{\infty} \Phi(a + bx) \phi(x) dx = \Phi \left( \frac{a}{\sqrt{1 + b^2}} \right) \]

(7)

Using Owen’s [3] results, Eq. (1) can be rewritten as

\[ \alpha = \Phi(USL) - \Phi(LSL) - [\Phi(USL) - BVN(USL, LSL, \rho) - BVN(LSL, USL, -\rho)] + [\Phi(LSL) - BVN(LSL, LSL, \rho) - BVN(LSL, -USL, -\rho)] \]

\[ = BVN(USL, LSL, \rho) + BVN(USL, -USL, -\rho) - BVN(LSL, LSL, \rho) - BVN(LSL, -USL, -\rho) \]

(8)

The probability of accepting a defective unit is the probability that \(Y\) falls in the acceptance region when \(X\) is in the rejection region. Hence, the probability of type II
error, \( \beta \), is

\[
\beta = P(LSL \leq Y \leq USL, \text{ when } X < LSL \text{ or } X > USL) \\
= \int \left[ \int \phi(y|x) dy \right] \phi(x) dx + \int \left[ \int \phi(y|x) dy \right] \phi(x) dx \\
= \int \left[ \int d\Phi \left( \frac{y - \rho x}{\sqrt{1 - \rho^2}} \right) \phi(x) dx \right] + \int \left[ \int d\Phi \left( \frac{y - \rho x}{\sqrt{1 - \rho^2}} \right) \phi(x) dx \right] \\
= \int \left[ \Phi \left( \frac{USL - \rho x}{\sqrt{1 - \rho^2}} \right) - \Phi \left( \frac{LSL - \rho x}{\sqrt{1 - \rho^2}} \right) \right] \phi(x) dx + \int \left[ \Phi \left( \frac{USL - \rho x}{\sqrt{1 - \rho^2}} \right) - \Phi \left( \frac{LSL - \rho x}{\sqrt{1 - \rho^2}} \right) \right] \phi(x) dx
\]

(9)

where \( LSL, \ USL, X, Y, \phi(x), \Phi(\cdot) \), and \( \phi(y|x) \) are the same as Eq. (3).

Using Owen’s [3] results, Eq. (9) can be rewritten as

\[
\beta = BVN(USL, LSL, \rho) + BVN(USL, -USL, -\rho) - BVN(LSL, LSL, \rho) - BVN(LSL, -USL, -\rho)
\]

(10)

Hence, total risk of making wrong decision is

\[
\alpha + \beta = 2[BVN(USL, LSL, \rho) + BVN(USL, -USL, -\rho) - BVN(LSL, LSL, \rho)]
\]

(11)

4. Risk Evaluation of Product

Mee and Owen [4] proposed the approximate formula for computing the bivariate normal cumulative probability. Assume that the standardized target value of product is zero. Consider the symmetric and asymmetric specification limits for computing the approximation of \( \alpha + \beta \). From Mee and Owen [4], Eq. (11) can be rewritten as the following three cases:

Case 1. \( LSL = -\Delta_2, USL = \Delta_1 \), the \( |\Delta_1| > |\Delta_2| \)

\[
\alpha + \beta = 2[\Phi(-\Delta_2) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 - A}{B} \right) + \Phi(-\Delta_2) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 + A}{B} \right) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 - C}{D} \right) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 - A}{B} \right)]
\]

(12)

where \( A = \rho \frac{\Phi(-\Delta_1)}{\Phi(-\Delta_2)}; \ B = \sqrt{1 + \Delta_2 \rho A - A^2}; \ C = -\rho \frac{\Phi(-\Delta_1)}{\Phi(-\Delta_2)}; \ D = \sqrt{1 - \Delta_2 \rho C - C^2} \).

Case 2. \( LSL = -\Delta_2, USL = \Delta_1 \), and \( |\Delta_2| > |\Delta_1| \)

\[
\alpha + \beta = 2[\Phi(-\Delta_2) \cdot \Phi\left( -\frac{\Delta_1 - A}{B} \right) + \Phi(-\Delta_2) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 - C}{D} \right) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta_1 - A}{B} \right)]
\]

(13)

where \( A = -\rho \frac{\Phi(-\Delta_1)}{\Phi(-\Delta_2)}; \ B = \sqrt{1 - \Delta_1 \rho A - A^2}; \ C = -\rho \frac{\Phi(-\Delta_1)}{\Phi(-\Delta_2)}; \ D = \sqrt{1 - \Delta_1 \rho C - C^2} \).

Case 3. \( LSL = -\Delta_2 \) and \( USL = \Delta_1 \)

\[
\alpha + \beta = 2[\Phi(-\Delta_2) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta - A}{B} \right) + \Phi(-\Delta_2) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta + A}{B} \right) - \Phi(-\Delta_1) \cdot \Phi\left( -\frac{\Delta - A}{B} \right) - 4\Phi(-\Delta) \cdot \Phi\left( -\frac{\Delta - A}{B} \right)]
\]

(14)

where \( A = \rho \frac{\Phi(-\Delta_2)}{\Phi(-\Delta_1)}; \ B = \sqrt{1 + \Delta_1 \rho A - A^2} \).

For Eqs. (12)–(14), the minimum total risk is zero, respectively. Now, we need to evaluate the maximum to-
One can set the search limit of $\Delta$, $\Delta_1$, and $\Delta_2$ and adopt the direct search method for obtaining the maximum total risk when the correlation coefficient $\rho$ is given. Tables 1–2 list the maximum total risk for symmetric and asymmetric specification limits. From Tables 1–2, we have the following conclusion: the larger the value of $\rho$ (the smaller the measurement error), the smaller maximum total risk. It is clear that the reduction of the variance of measurement error will increase the probability of making correct decisions.

5. Numerical Example

Assume that the relationship between the true value of $X$ and the observed value of $Y$ of product is probabilistic. The $X$ and $Y$ are dependent variables with positive correlation coefficient $\rho$. The joint probability density function of $(X, Y)$ is a standard bivariate normal distribution. From process data, the value of $\rho$ is estimated to be 0.3. That is, the variance of the measurement error is 0.91. In general, the customer defines the specification limits. Hence, we have that maximum total risk of wrong decisions is about 0.48255 for customer’s symmetric or asymmetric specification limits. If we can further decrease the operator or gauge error, then the measurement error will also decrease. We will have a smaller total risk under the given specification limits of customers.

6. Conclusions

In this paper, we have presented the modified Chandra’s [2] independent model that the true value and the observed value of product characteristic are dependent under the measurement error. For decreasing the maximum total risk of wrong decisions about product, we need to reduce the measurement error. It is also clear that the reduction of the variance of measurement error, will increase the probability of making correct decisions. Chandra [2] pointed out that ‘we need to compare the variance of operator reproducibility and the variance of gauge repeatability, in order to find the method of reducing the variance of measurement error.’

References


