Mass Transfer Modeling of Conjugated Graetz Problem in Multi-Pass Mass Exchangers with External Recycle

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Abstract

A new design of multi-pass mass exchanger with external recycle and the mass transfer in such a device have been investigated theoretically in this study. The analytical solutions were obtained by applying an orthogonal expansion technique associated with the eigenfunction expanding in terms of an extended power series. The influences of the design parameters (the subchannel height ratio, \( \beta \), and permeable membrane parameter, \( \gamma \)) and operating conditions (the recycle ratio, \( R \), and mass-transfer Graetz number, \( G_{zm} \)) on the outlet concentration and mass transfer rate were discussed and represented graphically. Comparing to the single-pass mass exchangers without external recycle, the mass transfer efficiency was improved by employing the multi-pass devices with external recycle. In economic sense, the optimal operating conditions, say \( R = 1, \beta_{ab} = \beta_{cd} = 1/3 \) and \( G_{zm} > 30 \) for \( \gamma = 10 \), were selected by considering both the mass transfer efficiency improvement and power consumption increment in this study.

Key Words: Mass Transfer, External Recycle, Multi-Pass Operations, Conjugated Graetz Problem

1. Introduction

The laminar forced convection heat transfer phenomena of a flowing fluid inside a bounded conduit under constant wall temperature or heat flux and with neglecting axial conduction was firstly analyzed by Graetz [1]. This kind of heat transfer problem is the well-known classical Graetz problem [2–5]. The extension of the classical Graetz problem for the low Prandtl number fluid, such as liquid metal, with concerning the axial conduction is called the extended Graetz problem [6–9]. Beyond the general concern for a single-stream problem, the multi-pass or multi-phase systems coupled through the mutual conditions at the boundaries are referred to the conjugated Graetz problems [10–13]. Analog to heat transfer problems, the analysis of mass transfer phenomena in a mass exchanger has also been reported in many studies [14,15]. Cooney et al. [14] investigated the mass transfer in hemodialyzers of parallel-plate and cylindrical type with constant dialysate concentration. A mathematical model for the mass transfer problem of parallel-plate and cylindrical-tube hemodialyzers was developed and solved by using separation of variables and Kummer’s equation. Ho and Tu [15] formulated a mathematical model of the mass-transfer conjugated Graetz problem of the multi-pass mass exchangers and demonstrated that the multi-pass design can improve the mass-transfer rate by comparing with that in an open-duct device.

As referred to many investigators, the heat or mass transfer rate can be enhanced by dividing an open-duct device into several subchannels [16,17] or reducing the channel size [18,19]. By those designs, the fluid velocity in the conduit can be increased and consequently the heat and mass transfer coefficient also be increased. Another strategy to improve device performance is to introduce the recycle concept on the devices. The external or internal recycle concept is widely
applied in industrial chemical engineering processes such as reactors [20], thermal diffusion column [21], drift-tube bubble column [22] and air-lift reactor [23]. The recycle operations may cause two conflicting effects on the mass transfer: the desirable effect of increasing fluid velocity, resulting in increasing mass transfer coefficient, and the undesirable effect of reducing the mass transfer driving force (concentration gradient). The mass-transfer efficiency improvement is achieved while the first effect overcomes the latter one. Therefore, by introducing the multi-pass design and recycle concept on a mass exchanger, the mass transfer rate improvement can be expected.

The purposes of this study are to design a new multi-pass mass exchanger with external recycle, to develop the mathematical formulation of a multi-pass mass exchanger and to find out the optimal design parameters and operating conditions. The mass transfer efficiency improvement and power consumption increment caused by introducing multi-pass design and recycle concept into a mass exchanger were calculated and discussed based on the single-pass devices without recycle. The theoretical predictions were presented graphically comparing to the single- and double-pass devices without recycle. The uniform wall concentration considered in this study is usually applied in membrane contactors operation such as hemodialysis [14] or gas absorption [24]. The improvement of mass-transfer efficiency in a hemodialyzer can shorten the amount of time a patient needs to spend on hemodialysis while that in a gas absorption membrane contactor can minimize the usage of expensive absorber. These are the main motivations in the present study to design a high mass-transfer efficiency mass exchanger.

2. Mass Transfer in Multi-Pass Mass Exchangers

A multi-pass mass exchanger was designed by inserting three ideal permeable membranes with negligible thickness \( \delta \ll \) into a parallel-plate with height \( W \), length \( L \), and infinite width \( B (B >> W) \) to divide an open duct into four parts, subchannels \( a, b, c \) and \( d \) with height \( W_a, W_b, W_c \) and \( W_d \), respectively. The subchannel height ratio is defined as \( \beta = W_a/W_b = W_d/W_c \). A fluid with volumetric flow rate \( V \) and concentration \( C_i \) was fed into the two inner subchannels as shown in Figure 1. The fluid entering the two outer subchannels at the other end mixed with the fluid exiting from the two inner subchannels.

![Figure 1. Multi-pass mass exchanger with external recycle.](image)
The fluid volumetric flow rate $RV$ and outlet concentration $CF$ exiting from the two inner subchannels occurred through the aid of convectional pumps. Three permeable membranes are inserted to divide an open-duct device into a multi-pass device and the solute can pass through the permeable membrane. The solute is firstly dissolved from the walls and then transported by the force convection along the flow direction as well as penetrates through the three idealized membranes from higher concentration side to lower one by diffusion. Moreover, the transmembrane pressure is kept zero during whole operation in this study and thus no solvent penetrates through the membrane.

The analysis of mass transfer in a multi-pass mass exchanger was simplified by assuming steady state, fully-developed laminar flow in each subchannel, neglecting the entrance length and longitudinal diffusion, ignoring the end effects and constant physical properties of fluid. The concentration polarization phenomenon on the membrane is also neglected by assuming that the solute can freely pass through the membrane. The mass balance equations and velocity distributions \[25\] in dimensionless form for a multi-pass mass exchanger with constant wall concentration can be written as:

\[
\frac{\partial^2 \psi_i(\eta_i, \xi)}{\partial \eta_i^2} = \frac{W_i}{L} \frac{\partial \psi_i(\eta_i, \xi)}{\partial \xi}, \quad i = a, b, c, d \tag{1}
\]

\[v_i(\eta_i) = -\varphi_i(6\eta_i - \eta_i^2), \quad 0 \leq \eta_i \leq 1, \quad i = a, d
\tag{2a}
\]

\[v_i(\eta_i) = \varphi_i(6\eta_i - 6\eta_i^2), \quad 0 \leq \eta_i \leq 1, \quad i = b, c
\tag{2b}
\]

in which

\[\varphi_a = [(R + 1)V \vert W_s B], \quad \varphi_v = [V \vert W_s B], \quad \varphi_c = [V \vert W_s B],\]

\[\varphi_d = [(R + 1)V \vert W_s B], \quad \xi = z / L, \quad \eta_i = x_i / W_s, \quad i = a, d
\tag{3}
\]

\[\psi_i = (C_i - C_f) / C_j - C_i, \quad 0 = 1 - \psi_i, \quad G_\psi = 2VW / DBL
\]

The corresponding boundary conditions are

\[\psi_a(0, \xi) = 0
\tag{4}
\]

\[-\frac{\partial \psi_d(1, \xi)}{\partial \eta_d} = \frac{W_d}{W_s} \frac{\partial \psi_d(1, \xi)}{\partial \eta_d}
\tag{5}
\]

\[-\frac{\partial \psi_a(1, \xi)}{\partial \eta_a} = -\frac{W_a}{W_s} [\psi_a(1, \xi) - \psi_s(1, \xi)]
\tag{6}
\]

where $\gamma$ is the permeable membrane parameter, $\gamma = \varepsilon (W / \delta)$. The boundary conditions of Eqs. (5), (7) and (9) show the equal mass flux between each two subchannels on the permeable membranes while the boundary conditions of Eqs. (6), (8) and (10) on the permeable membranes are derived from the modified Fick’s law \[26\]. The dimensionless outlet concentration is

\[\theta_f = 1 - \psi_f = \frac{C_f - C_j}{C_f - C_i}
\tag{12}
\]

By following the same calculation procedure performed in the previous work \[15\], one can separate variables in the form

\[\psi_i(\eta_i, \xi) = \sum_{m=0} S_{i, m} F_{i, m}(\eta_i) G_m(\xi)
\tag{13}
\]

Substituting Eq. (13) into Eq. (1) results

\[G_m(\xi) = e^{-\xi m (1 - \xi)} \quad 0 \leq \xi \leq 1
\tag{14}
\]

\[F_{i, m}(\eta_i) - \frac{\lambda_m W_s^2 \varphi_i(\eta_i)}{LD} F_{i, m}(\eta_i) = 0
\tag{15}
\]

and the corresponding boundary conditions in Eqs. (4)–(11) can be rewritten as

\[F_{a, m}(0) = 0
\tag{16}
\]

\[S_{i, m} F_{i, m}(1) = -\frac{W_d}{W_s} S_{i, m} F_{i, m}(1)
\tag{17}
\]
\[ S_{a,m}F'_{a,m}(l) = -\frac{\gamma W}{W} [S_{a,m}F_{a,m}(l) - S_{b,m}F_{b,m}(l)] \quad (18) \]
\[ S_{b,m}F'_{b,m}(0) = -\frac{W_b}{W} S_{c,m}F'_{c,m}(0) \quad (19) \]
\[ S_{b,m}F'_{b,m}(0) = -\frac{\gamma W}{W} [S_{b,m}F_{b,m}(0) - S_{c,m}F_{c,m}(0)] \quad (20) \]
\[ S_{c,m}F'_{c,m}(l) = -\frac{W_c}{W} S_{d,m}F'_{d,m}(l) \quad (21) \]
\[ S_{c,m}F'_{c,m}(l) = -\frac{\gamma W}{W} [S_{c,m}F_{c,m}(l) - S_{d,m}F_{d,m}(l)] \quad (22) \]
\[ F_{d,m}(0) = 0 \quad (23) \]

The eigenfunctions \( F_{a,m}(\eta_a), F_{b,m}(\eta_b), F_{c,m}(\eta_c) \) and \( F_{d,m}(\eta_d) \) were assumed to be polynomials to avoid the generality loss and with the aid of the boundary conditions, one obtains

\[ F_{a,m}(\eta_a) = \sum_{n=0}^{\infty} p_{a,m,n} \eta_a^n, \quad p_{a,m,0} = 0, \quad p_{a,m,1} = 1 \quad (selected) \quad (24) \]
\[ F_{b,m}(\eta_b) = \sum_{n=0}^{\infty} q_{b,m,n} \eta_b^n, \quad q_{b,m,0} = 0, \quad q_{b,m,1} = 1 \quad (selected) \quad (25) \]
\[ F_{c,m}(\eta_c) = \sum_{n=0}^{\infty} r_{c,m,n} \eta_c^n, \quad r_{c,m,0} = 0, \quad r_{c,m,1} = 1 \quad (selected) \quad (26) \]
\[ F_{d,m}(\eta_d) = \sum_{n=0}^{\infty} s_{d,m,n} \eta_d^n, \quad s_{d,m,0} = 0, \quad s_{d,m,1} = 1 \quad (selected) \quad (27) \]

Combining Eqs. (17)–(22), one may get the following equations and the eigenvalues (\( \lambda_1, \lambda_2, \ldots, \lambda_{m-1} \)) can be calculated from Eqs. (28)–(30),

\[ S_{a,m} = \frac{\gamma W}{\gamma W} [S_{a,m}F_{a,m}(l) + W_{a,m}(l)] \quad (28) \]
\[ S_{b,m} = \frac{\gamma W}{\gamma W} [S_{b,m}F_{b,m}(0) + W_{b,m}(0)] \quad (29) \]
\[ S_{c,m} = \frac{\gamma W}{\gamma W} [S_{c,m}F_{c,m}(l) + W_{c,m}(l)] \quad (30) \]

The expansion coefficients, \( S_{a,m}, S_{b,m}, S_{c,m}, S_{d,m} \) and the associated eigenfunctions \( F_{a,m}, F_{b,m}, F_{c,m} \) and \( F_{d,m} \) can be found while all of the eigenvalues \( (\lambda_m) \) calculated from Eqs. (28)–(30). The average dimensionless outlet concentration \( \psi_{F,a,b} \) or \( \psi_{F,c,d} \) can be determined by

\[ \psi_{F,a,b} = \frac{-\int_0^1 \psi_{F,b} \psi_{a,b}(\eta_a,0) d\eta_b}{(R + 1) V} \quad (31) \]

or

\[ \psi_{F,c,d} = \frac{-\int_0^1 \psi_{F,c} \psi_{c,d}(\eta_c,0) d\eta_d}{(R + 1) V} \quad (32) \]

Moreover, the mixed dimensionless concentration at the end of the two outer subchannels are calculated by

\[ \psi_{L,a,b} = \frac{R}{R + 1} \int_0^1 \psi_{L,b} \psi_{a,b}(\eta_a,0) d\eta_b - \int_0^1 \psi_{L,b} \psi_{a,b}(\eta_a,1) d\eta_b \]

\[ = \frac{-2RW}{W_j (R + 1) G_{z_m}} \sum_{m=1}^{\infty} \frac{e^{-\lambda_m} S_{a,m}}{\lambda_m} [F_{a,m}(l) - F_{a,m}(0)] \quad (33) \]

or

\[ \psi_{L,c,d} = \frac{R}{R + 1} \int_0^1 \psi_{L,d} \psi_{c,d}(\eta_c,0) d\eta_d - \int_0^1 \psi_{L,d} \psi_{c,d}(\eta_c,1) d\eta_d \]

\[ = \frac{-2RW}{W_j (R + 1) G_{z_m}} \sum_{m=1}^{\infty} \frac{e^{-\lambda_m} S_{c,m}}{\lambda_m} [F_{c,m}(l) - F_{c,m}(0)] \quad (34) \]

### 3. Mass Transfer in Single- and Double-Pass Mass Exchangers

The single- and double-pass mass exchangers without external recycle are shown in Figures 2(a) and 2(b), respectively. Following the same calculating procedure performed in the previous section, the dimensionless outlet concentrations for the double-pass devices (\( \theta_P \) and \( \theta_Q \))
gle-pass devices \((\theta_{0,F})\) were obtained in terms of eigenvalues \((\lambda_m \text{ and } \lambda_{0,m})\), expansion coefficients \((S_{a,m}, S_{b,m} \text{ and } S_{0,m})\), eigenfunctions \((F_{a,m}(\eta_a), F_{b,m}(\eta_b) \text{ and } F_{0,m}(\eta_0))\), mass-transfer Graetz number \((G_z_m)\) and subchannel height ratio \((\beta)\). The results are

\[
\theta_F = 1 - \psi_F = \frac{1}{G_z_m} \left[ \sum_{m=0}^{\infty} \frac{(1 - e^{-\lambda_m})}{\lambda_m W_a} S_{a,m} F_{a,m}'(0) \right. \\
\left. + \sum_{m=0}^{\infty} \frac{(1 - e^{-\lambda_{0,m}})}{\lambda_{0,m} W_0} S_{0,m} F_{0,m}'(0) \right]
\]  \(\text{(35)}\)

and

\[
\theta_{0,F} = 1 - \psi_{0,F} = \frac{1}{G_z_m} \left[ \sum_{m=0}^{\infty} \frac{(1 - e^{-\lambda_{0,m}})}{\lambda_{0,m} W_0} S_{0,m} F_{0,m}'(0) \right. \\
\left. + \sum_{m=0}^{\infty} \frac{(1 - e^{-\lambda_{0,m}})}{\lambda_{0,m} W_0} S_{0,m} F_{0,m}'(1) \right]
\]  \(\text{(36)}\)

4. Mass Transfer Efficiency

Based on the maximum concentration difference, the average mass transfer coefficient was defined as

\[
N = \overline{k}_m (2BL) (C_i - C_f)
\]  \(\text{(37)}\)

By using the overall mass balance, the average mass transfer coefficient can be determined by

\[
\overline{k}_m (2BL) (C_i - C_f) = 2V (C_f - C_i)
\]  \(\text{(38)}\)

or

\[
\overline{k}_m = \frac{2V}{2BL} \left( \frac{C_f - C_i}{C_i - C_f} \right) = \frac{2V}{2BL} (1 - \psi_F)
\]  \(\text{(39)}\)

*Figure 2. Single- and double-pass parallel-plate mass exchangers.*
The average Sherwood number was defined as

$$\overline{Sh} = \frac{kW}{D}$$  \hspace{1cm} (40)

The substitution of Eq. (39) into Eq. (40) gives

$$\overline{Sh} = \frac{kW}{D} = \frac{2W}{2DBL} (1-\gamma_F)$$
$$= 0.5Gz_a (1-\gamma_F) = 0.5Gz_{a} \theta_F$$  \hspace{1cm} (41)

for double-pass devices while Eq. (42) is for single-pass devices

$$\overline{Sh} = \frac{k_{a,0}W}{D} = \frac{2W}{2DBL} (1-\gamma_{0,F})$$
$$= 0.5Gz_a (1-\gamma_{0,F}) = 0.5Gz_{a,0} \theta_{0,F}$$  \hspace{1cm} (42)

Based on a single-pass device with same working dimensions and without external recycle, the mass transfer efficiency improvement, $I_m$, by employing a multi-pass device is best illustrated by calculating the percentage increase in mass-transfer rate, as shown in following

$$I_m = \frac{\overline{Sh} - \overline{Sh}_0}{\overline{Sh}_0} \times 100\% = \frac{\Psi_{0,F} - \Psi_{F}}{1-\Psi_{0,F}} \times 100\%$$  \hspace{1cm} (43)

5. Power Consumption Increment

The friction loss on the conduit surface can be estimated by

$$\ell_{W_f} = \frac{2f \nabla^2 L}{D_c}$$  \hspace{1cm} (44)

where $\nabla$ and $D_c$ refer to the average velocities of fluid in the conduits and the equivalent diameters of the conduits, respectively. For the laminar flow and parallel-plate conduits, the friction factor $f$ in the Eq. (44) is a function of Reynolds number, Re, as $f = 24/Re$. The equivalent diameters of the conduits for the single- and multi-pass device in this study are

$$D_{c,0} = 2W$$  \hspace{1cm} (45)

and

$$D_{c,b} = 2W_b$$
$$D_{c,c} = 2W_c$$
$$D_{c,d} = 2W_d$$  \hspace{1cm} (46)

respectively. The power consumption increment, $I_p$, can be defined as

$$I_p = \frac{(P_a + P_b + P_c + P_d) - P_0}{P_0}$$
$$= \frac{\ell (R+1)(\ell_{W_{f,a}} + \ell_{W_{f,b}} + \ell_{W_{f,c}} + (R+1)\ell_{W_{f,d}}) - 2\ell_{W_{f,0}}}{2\ell_{W_{f,0}}}$$  \hspace{1cm} (47)

where the $P_0$ is the power consumption of single-pass devices and the $P_a$, $P_b$, $P_c$ and $P_d$ are the power consumptions in the subchannel $a$, $b$, $c$ and $d$ of multi-pass devices, respectively. Substituting Eqs. (44)–(46) and the average velocity into Eq. (47) gives

$$I_p = \frac{(R+1)^2}{2} \left( \frac{W}{W_a} + 1 - \frac{1}{2} \frac{W}{W_b} \right)$$  \hspace{1cm} (48)

The calculating procedure is shown in Appendix.

6. Results and Discussions

The mass transfer phenomenon of multi-pass mass exchangers with external recycle and constant wall concentration has been developed and solved by the orthogonal expansion technique and eigenfunctions expanding in terms of an extended power series. The eigenfunctions $F_{a,0}(\eta_a)$, $F_{b,0}(\eta_b)$, $F_{c,0}(\eta_c)$ and $F_{d,0}(\eta_d)$ were assumed to be polynomials, as shown in Eqs. (24)–(27), to avoid the generality loss. The convergence of the power series in Eqs. (24)–(27) with $n = 20$ and $25$ with $\beta_{a,b} = \beta_{c,d} = 1/3$, $R = 1$ and $\gamma = 1$ was shown in Table 1. It is clearly seen from the Table 1 that the power series agrees reasonably well with the terms of $n = 20$. Therefore, the power series with such selected terms was employed in this study. Two conflict effects are created by employing the recycle concept to the multi-pass mass exchangers: the desired effect is the convection mass transfer coefficient decreasing due to the fluid velocity increasing in the conduit, and the undesired one is the mass transfer driving force (concentration gradient) decreasing. While the first effect compensates the last one, the mass transfer efficiency improvement by employing the recycle concept to the multi-pass mass exchangers can be expected.

The average dimensionless outlet concentration, $\theta_{f,0}$,
of the multi-pass mass exchangers was calculated by Eqs. (31) and (32) and the calculating results are illustrated in Figure 3. The large permeable membrane parameter, \( \gamma \), means that the solute is more easily pass through the permeable membrane. Hence, the average dimensionless outlet concentration increases with increasing the permeable membrane parameter \( \gamma \) as shown in Figure 3. The influence of subchannel height ratio, \( \beta_{ab} \) (or \( \beta_{cd} \)) on \( \theta_F \) is also presented in the Figure 3 and the results show that the higher \( \theta_F \) can be obtained while the \( \beta_{ab} \) (or \( \beta_{cd} \)) decreases. The reason why the smaller value of \( \beta_{ab} \) (or \( \beta_{cd} \)) has higher outlet concentration is that due to the mass transfer mainly occurs on the two outer walls while the \( \beta_{ab} \) (or \( \beta_{cd} \)) decreases, i.e. the height of outer subchannel decreases, the mean velocity and the mass-transfer coefficient in the outer subchannel consequently increase and thus the higher outlet concentration is obtained. Moreover, because of the fluid residence time in the device decreasing with increasing the large mass-transfer Graetz number, \( G_{zm} \) (increasing volumetric flow rate or decreasing conduit length), the dimensionless outlet concentration decreases with increasing \( G_{zm} \). As shown in Figure 3, the outlet concentration of a single-pass device \( \theta_{0,F} \) is higher than that in the double- and multi-pass devices in small mass-transfer Graetz number, but it is lower when the large mass-transfer Graetz number is operated. The mass transfer by diffusion is more important than that by force convection, and hence a single-pass device has the longer residence time to transfer than that in double- and multi-pass devices due to the lower mean velocity for the lower \( G_{zm} \) (low volumetric flow rate or long conduit length). Thus the higher outlet concentration is obtained in the single-pass device. In contrast, in the higher \( G_{zm} \) (high volumetric flow rate or short conduit length), the influence of force convection on the mass transfer is more significant than that of diffusion, and thus, the double- and multi-pass devices achieve a higher outlet concentration than that in single-pass devices due to the higher mean velocity, i.e. the larger mass-transfer coefficients. The average Sherwood number versus mass-transfer Graetz number with the subchannel height ratio as a parameter for \( \gamma = 5 \) is shown in Figure 4. As shown in Eq. (41), the \( \overline{Sh} \) is direct proportion to \( G_{zm} \theta_F \) which can be imaged as the total amount of solute obtained at outlet. Therefore, although the \( \theta_F \) de-

<table>
<thead>
<tr>
<th>( G_z )</th>
<th>( n )</th>
<th>( \lambda_m )</th>
<th>( S_{a,m} )</th>
<th>( S_{b,m} )</th>
<th>( S_{c,m} )</th>
<th>( S_{d,m} )</th>
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creases with increasing $Gzm$ as shown in Figure 3, the $\tilde{Sh}$ (or the product of $Gzm$ and $\theta_F$) increases with increasing $Gzm$ as shown in Figure 4. Meanwhile, Figure 4 also shows that the effect of $\beta_{ab}$ (or $\beta_{cd}$) is more sensitive when $Gzm > 10$. It can be concluded from Figure 4 that the average Sherwood numbers increases with decreasing the recycle ratio and subchannel height ratio. Figure 5 illustrates that the average Sherwood number increases with increasing permeable membrane parameter but decreases with increasing recycle ratio for a fixed subchannel height ratio, $\beta_{ab} = \beta_{cd} = 1/3$. Observing from Eqs. (41) and (42), the average Sherwood number proportions to the $Gzm$ and, consequently, the average Sherwood number increases with increasing $Gzm$ as indicated in Figures 4 and 5.

The mass transfer efficiency improvement, $I_m$, for the multi-pass mass exchangers with external recycle was defined as the percentage increase in mass-transfer rate based on that obtained by employing the single-pass devices with same working dimensions and without external recycle as shown in Eq. (43). The calculating results of $I_m$ for $\gamma = 5$ are shown in Table 2. Table 2 indicates that the mass transfer efficiency improvement increases with decreasing with increasing $R$ and $\beta$. Moreover, because of the mass transfer efficiency improvement, $I_m$, is direct proportion to the $(\tilde{Sh} - \tilde{Sh}_0)$ and $(\tilde{Sh} - \tilde{Sh}_0)$ increases with increasing $Gzm$, observing from Figures 4 and 5, hence the $I_m$ increases with increasing $Gzm$, as indicated in Table 2, although the $\theta_F$ decreases with increasing $Gzm$. It should be mentioned that the negative signs in the Table 2 shows that the mass transfer rate by employing a single-pass device without recycle is greater than that in a multi-pass devices with external recycle. Hence, in this case, a single-pass device is alternative rather than using a multi-pass device with external recycle. In contrast, A multi-pass device is more efficient than single-pass device for the cases with positive mass-transfer efficiency improvement $I_m$ value, as shown in the Table 2. The multi-pass design and recycle operation produce two conflict effects on the mass-transfer effi-

![Figure 4. Average Sherwood number vs. $Gzm$ with $\beta_{ab}$ (or $\beta_{cd}$) as a parameter for $\gamma = 5$.](image)

![Figure 5. Average Sherwood number vs. $Gzm$ with recycle ratio as a parameter for $\beta_{ab} = \beta_{cd} = 1/3$.](image)
The mass transfer efficiency improvement with the channel height ratio as a parameter $\gamma = 5$

<table>
<thead>
<tr>
<th>$G_{zm}$ (%)</th>
<th>$\beta_{ab} = \beta_{cd} = 1/3$</th>
<th>$\beta_{ab} = \beta_{cd} = 1$</th>
<th>$\beta_{ab} = \beta_{cd} = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{zm} = 1$</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
</tr>
<tr>
<td>$G_{zm} = 10$</td>
<td>71.96</td>
<td>47.86</td>
<td>28.00</td>
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<td>$G_{zm} = 100$</td>
<td>303.42</td>
<td>133.73</td>
<td>66.22</td>
</tr>
<tr>
<td>$G_{zm} = 1000$</td>
<td>387.47</td>
<td>153.63</td>
<td>73.49</td>
</tr>
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</table>

$R = 1$

$G_{zm} = 1$ | $\beta_{ab} = \beta_{cd} = 1/3$ | $\beta_{ab} = \beta_{cd} = 1$ | $\beta_{ab} = \beta_{cd} = 3$ |
<table>
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<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{zm} = 10$</td>
<td>0.05</td>
<td>-0.02</td>
<td>-0.39</td>
</tr>
<tr>
<td>$G_{zm} = 100$</td>
<td>60.97</td>
<td>34.64</td>
<td>15.34</td>
</tr>
<tr>
<td>$G_{zm} = 1000$</td>
<td>284.58</td>
<td>124.17</td>
<td>58.99</td>
</tr>
<tr>
<td>$G_{zm} = 10000$</td>
<td>381.39</td>
<td>148.93</td>
<td>68.88</td>
</tr>
</tbody>
</table>

$R = 5$

<table>
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<tr>
<th>$G_{zm}$ (%)</th>
<th>$\beta_{ab} = \beta_{cd} = 1/3$</th>
<th>$\beta_{ab} = \beta_{cd} = 1$</th>
<th>$\beta_{ab} = \beta_{cd} = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{zm} = 1$</td>
<td>0.036</td>
<td>-0.36</td>
<td>-1.40</td>
</tr>
<tr>
<td>$G_{zm} = 10$</td>
<td>56.16</td>
<td>29.93</td>
<td>11.09</td>
</tr>
<tr>
<td>$G_{zm} = 100$</td>
<td>277.98</td>
<td>120.31</td>
<td>55.87</td>
</tr>
<tr>
<td>$G_{zm} = 1000$</td>
<td>378.35</td>
<td>146.26</td>
<td>66.33</td>
</tr>
</tbody>
</table>

Although introducing the multi-pass design and recycle concept to a mass exchanger can improve the mass transfer efficiency, the power consumption is also increasing by employing such designs. Base on the following working dimensions: $L = 1.2 \, m$, $W = 0.04 \, m$, $B = 0.2 \, m$, $V = 1 \times 10^{-5} \, m^3/s$, $\mu = 8.94 \times 10^{-4} \, kg/ms$, $\rho = 997.08 \, kg/m^3$, the power consumption of a single-pass device is $P_0 = 2V\rho (\text{losses}) = 2.68 \times 10^{-7} \, J/s$. In order to confirm the laminar flow, the Reynolds number was estimated as

$$\text{Re} = \frac{\rho (R + 1) D \mu}{BW} = 2\rho (R + 1)V \frac{B \mu}{B W} \Gamma$$

where the $D_e = 2W$. The minimum and maximum Reynolds numbers in this study are $\text{Re}_{\text{min}} = 223.1$ and $\text{Re}_{\text{max}} = 669.2$ for $R = 1$ and 5, respectively. Therefore, all the systems discussed in this study are laminar flow, say $\text{Re} < 2,100$. We consider the system with developing region in front of fully-developed region as an illustration. With $\text{Re} = 669.2$, $W_0 = 0.005$, the entrance length $L_e$ [25] is obtained as 0.23 m and the $L_e/L$ is 0.2. For comparison simplicity, only the friction loss in the devices was considered in the power consumption increment of multi-pass operations. A pressure drop caused by a joint, a diversion or a bending of a tube is neglected in the present study. As indicating from Eq. (48), the power consumption increment, $I_p$, does not depend on Graetz number but proportions to the recycle ratio. The calculating results for $I_p$ are presented in Table 2. As shown in Table 2, the $I_p$ increases while the recycle ratio $R$ increases and moves away from 1, especially for $\beta > 1$. The highest power consumption increment, as shown in Table 2, is 9452.04 while a multi-pass device is operated at $R = 5$ and $\beta_{ab} = \beta_{cd} = 3$. Fortunately, the power consumption under these design and operating parameters are still small even for $\beta_{ab} = \beta_{cd} = 3$ with $R = 5$, say $P = 2.53 \times 10^{-3} \, J/s$. In economic sense, the optimal operating conditions for the multi-pass mass exchangers with external recycle are to consider both the mass-transfer efficiency improvement and the power consumption increment, $I_p/I_L$, and illustrated in Figure 6 for $\gamma = 10$. The positive values of $I_p/I_L$ mean that the mass transfer rate of multi-pass mass exchangers with external recycle are greater than that of single-pass devices with-
7. Conclusion

The mass transfer mathematical formulation of the new multi-pass mass exchangers with external recycle has been developed and solved in this study. The analytical solutions were obtained by using an orthogonal expansion technique associated with the eigenfunction expanding in power series. Two conflicts effects are created by introducing the multi-pass design and recycle operation: mass transfer coefficient increasing and driving force decreasing. While the first effect compensates the last one, the mass transfer efficiency can be improved. The calculating results show that the mass transfer rate of the multi-pass devices with external recycle increases with increasing mass-transfer Graetz number and permeable membrane parameter but decreases with increasing recycle ratio and the subchannel height ratio. Both the outlet concentration and average Sherwood number increase with the increasing permeable membrane parameter. The existence of membrane still has a positive influence on the mass-transfer efficiency enhancement for larger mass-transfer Graetz number, as confirmed from Figures 3 and 5. In economic sense, the optimal operating conditions by considering in this study are as indicated in Figure 6. The values of the operating conditions (Gzm and R) and design parameters (β and γ) selected in this study are illustrated to discuss the tendencies of these parameters on the mass transfer in the multi-pass mass exchangers and thus, only some specific values of these parameters have been used to simulate the mass transfer rate. Moreover, due to the mass transfer equations were derived under the limitations of laminar flow and steady state and hence, this theoretical model can be used to simulate the mass transfer rate while the operating conditions satisfy these limitations.

[Diagram: Figure 6. The values of I_m/I_p vs. Gzm with β_ab (or β_cd) as a parameter for γ = 10.]

Table 3. The power consumption increment with R and β_ab (or β_cd) as parameters

<table>
<thead>
<tr>
<th>R</th>
<th>β_ab = β_cd = 1/3</th>
<th>β_ab = β_cd = 1</th>
<th>β_ab = β_cd = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>84</td>
<td>79</td>
<td>577</td>
</tr>
<tr>
<td>1</td>
<td>293</td>
<td>159</td>
<td>1032</td>
</tr>
<tr>
<td>3</td>
<td>2455</td>
<td>799</td>
<td>4180</td>
</tr>
<tr>
<td>5</td>
<td>6740</td>
<td>1951</td>
<td>9452</td>
</tr>
</tbody>
</table>

Acknowledgement

The authors wish to thank the National Science Council of the Republic of China for its financial support.

Nomenclature

- B: conduit width, m
- C: concentration in the stream, mol/m³
- D: ordinary diffusion coefficient, m²/s
- D_e: equivalent diameter of the conduit, m
- F_m: eigenfunction associated with eigenvalue λ_m
- f: friction factor
- Gzm: mass-transfer Graetz Number, 2VW/DBL
- I_m: improvement of mass transfer, defined by Eq. (43)
- I_p: power consumption increment, defined by Eq. (48)
- k_m: average convection mass-transfer coefficient, m/s
- L: conduit length, m
Then, the friction loss of the single-pass devices can be obtained by taking the average velocity $\bar{v}_0 = 2V / BW$ and the equivalent diameter $D_{e,0} = 2W$ into the Eq. (A1)

$$\ell_{w_{f,0}} = \frac{24 \mu VL}{\rho BW^3} \quad (A2)$$

and the power consumption of single-pass devices is

$$P_0 = (2V \rho) \ell_{w_{f,0}} = \frac{48 \mu V^2 L}{BW^3} \quad (A3)$$

Similarly, the power consumptions of the fluid in the subchannel $a, b, c$ and $d$ of multi-pass devices are

$$P_a = [(R + 1)V \rho] \ell_{w_{f,a}} = \frac{24 \mu (R + 1)^2 V^2 L}{BW_a^3} \quad (A4)$$

$$P_b = (V \rho) \ell_{w_{f,b}} = \frac{24 \mu V^2 L}{BW_b^3} \quad (A5)$$

$$P_c = (V \rho) \ell_{w_{f,c}} = \frac{24 \mu V^2 L}{BW_c^3} \quad (A6)$$

and

$$P_d = [(R + 1)V \rho] \ell_{w_{f,d}} = \frac{24 \mu (R + 1)^2 V^2 L}{BW_d^3} \quad (A7)$$

Substituting Eqs. (A3)–(A7) into the Eq. (47) results

$$I_p = \frac{\ell_{w_{f,a}} + \ell_{w_{f,b}} + \ell_{w_{f,c}} + (R + 1)\ell_{w_{f,d}} - 2\ell_{w_{f,0}}}{2\ell_{w_{f,0}}}
= \frac{(R + 1)^2}{4} \left( \frac{W_a}{W_a} \right)^3 + \frac{1}{4} \left( \frac{W_b}{W_a} \right)^3 + \frac{1}{4} \left( \frac{W_c}{W_a} \right)^3 + \frac{(R + 1)^2}{4} \left( \frac{W_d}{W_a} \right)^3 \quad (A8)$$

Because of the $\beta_{ab} = \beta_{cd}$, i.e. $W_a = W_d$ and $W_b = W_c$ in this study, thus the Eq. (A8) can be reduced to

$$I_p = \frac{(R + 1)^2}{2} \left( \frac{W_a}{W_a} \right)^3 + \frac{1}{2} \left( \frac{W_b}{W_a} \right)^3 \quad (48)$$

References

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