Level-cut Approaches of First and Second Kind for Unique Solution Design in Fuzzy Engineering Optimization Problems

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Abstract

Two single level-cut approaches of the first and second kind for obtaining the unique compromise design in solving nonlinear optimum engineering design problems with fuzzy resources have been developed and presented in this paper. The conventional standard level-cuts method has been discussed for inspiring the proposed novel formulation consequently. The proposed strategies with the illustrative design examples indicate that the unique design as well as corresponding optimum level-cut value can be guaranteed obtained. Additionally, two wide-applicable linear or nonlinear membership functions of objective functions are presented depending on the practical situations of design tasks. The proposed level-cut approaches have been shown easy formulation and successfully employed to large-scaled structural design problems by sequential quadratic programming (SQP) technique combined with the finite element analysis.

Key Words: Level-cut Approach, Fuzzy Nonlinear Optimization, Engineering Design, Structural Optimization.

1. Introduction

The science and engineering in real-world problems are often not deterministic or non-crisp as people recognized. Fuzzy set theory [1] was a recent progress of describing certain non-crisp information with fuzziness arising in problems; since then, many fields ranging from sciences to industrial, medical and financial applications had applied it successfully. From the point of view of engineering, most applications and developments with fuzzy theory belong to the category of measurement, manufacturing and control behavior. However, literatures reported engineering designs and their applications with fuzzy logic are uncommon in dealing with the fuzziness existing

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of solutions are obtained by setting different level-cut value of \( \alpha \). The unique design level, indicated as \( \alpha^* \), may be obtained by minimizing an additional composing cost function with difficulties. Rao [7] applied the same level-cuts method to design a four-bar mechanism for function generating problem. He mentioned the difficulty of defining \( \alpha^* \) in original fuzzy 6++ problems is a nature reflection of the imprecision on the design problem; however, no formulation presented in the paper to achieve the unique value of \( \alpha^* \). Rao further proposed \( \lambda \)-formulation for achieving the unique design that is workable for fuzzy multi-objective optimization problems [8,9]. Yeh and Hsu [10] followed the framework of Wang et al. [6] under different design level of \( \alpha \) obtaining the optimum design level while the total cost is based on the failure possibility instead of the membership value of satisfaction. Xu [11] proposed bound search method of the 2nd phase optimization for obtaining the particular \( \alpha^* \) associated with the optimum point \( X^* \) by maximizing the established nonlinear fuzzy goal membership function \( \mu_c(X) \).

It can be summarized from the published literatures [6–11] for FNLP including fuzzy constraints that level-cuts method is accepted as the common solution approach for the problems with fuzzy resources. The preferred final design is obtained on the basis of predetermining cutting level of \( \alpha \) value. However, this predetermining cutting level is difficult to obtain, at the end of the solution process, the designer hardly achieves a definite and unique optimum result. Although Yeh and Hsu [10] presented unique optimum solution obtained from a certain cost function initiated by Wang et al., the determination of this suitable cost function is not straightforward. Although Rao [7] proposed \( \lambda \)-formulation for both the objectives and constraints are fuzzy that gives a unique solution; if only the constraints are fuzzy, further research work is needed in determining the optimum design by considering higher design level and other factors. Xu [11] proposed a second phase procedure called bound search method with fuzzy goal membership function to compute the unique solution. Both concepts in Rao [7] and Xu [11] are valuable for inspiring us re-considering about the \( \alpha \)-cuts method for efficiently achieve a unique optimum final solution.

In the start of this paper, we introduce general solution approaches initiated from FLP for nonlinear fuzzy resources problems of FNLP using level-cuts method. The inspiration of conventional level-cuts methods is discussed for the development of the presenting two level-cut approaches achieving the unique final design. The presenting level-cut approach of the first kind has been verified, with illustrative examples, to be the same as the level-cut approach of the second kind. Because the proposed approaches in this paper do not need extra factors [7] and functions of iterating bound search computation [11] to achieve the unique optimum design, therefore, the presenting approaches are convenient and fashionable in applications. Commonly used planar convex problems of three-bar and ten-bar truss with triangular fuzzy allowable range were presented for illustrating the proposed level-cuts methods. For further investigate the applicability of the proposed strategy, we apply it to a relatively complicated problem of 25-bar space truss design with two loading conditions. All design problems are solved by sequential quadratic programming with finite element analysis. The optimum design level and unique final design with linear or nonlinear fuzzy descriptions in design tasks are compared with crisp design. The mathematical formulations and computational algorithm are clearly given in the paper with the well illustrative examples.

2. Nonlinear Optimization with Fuzzy Constraints

The basic concepts and procedures of conventional linear programming with fuzzy constraints (FLP) [12] can be applied to nonlinear programming problems with fuzzy inequality constraints (FNLP). The general model of a nonlinear programming problem with fuzzy resources can be formulated as:

\[
\begin{align*}
\text{Find } & X = [x_1, x_2, \ldots, x_n]^T \\
\text{Min } & f(X) \\
\text{s.t. } & g_i(X) \leq \bar{b}_i, \ i = 1, 2, \ldots, m \\
& X^L \leq X \leq X^U
\end{align*}
\]

where the objective function and the \( i \)-th inequality constrained function are indicated as \( f(X) \) and \( g_i(X) \), respectively. \( X^L \) and \( X^U \) represent the lower bound and upper bound of design variables, respectively. The fuzzy number \( \bar{b}_i \), \( \forall_i \), are in the fuzzy region of \( \{b_i, b_i + p_i\} \) with given fuzzy tolerance \( p_i \). Assume that the fuzzy tolerance \( p_i \) for the \( i \)-th fuzzy constraint is known, then \( \bar{b}_i \) will be equivalent to \( (b_i + \theta p_i) \), \( \forall_i \), where \( \theta \) is in \([0, 1]\). In this case, a fuzzy constraints problem is transformed to be a crisp parametric programming problem. The following section, we summarized several level-cuts tech-
tiques for FLP problems that are applied to FNLP problems.

Verdegay’s Approach: \(\alpha\)-cuts Method

For dealing with Eqs. (1) and (2), Verdegay \[13\] considered that if the membership function of the fuzzy constraints (shown in Figure 1) has the following form:

\[\mu_{\alpha}(X) = \begin{cases} 
1 & \text{if } g_i(X) < b_i \\
1 - \frac{g_i(X) - b_i}{p_i} & \text{if } b_i \leq g_i(X) \leq b_i + p_i \\
0 & \text{if } g_i(X) > b_i + p_i
\end{cases}\]  

(3)

Simultaneously, the membership functions of \(\mu_{\alpha}(X)\), \(\forall \alpha\), are continuous and monotonic functions, and trade-off between those fuzzy constraints are allowed; then Eq. (1) and (2) is equivalent to the following formulation:

\[
\begin{align*}
\min & \quad f(X) \\
\text{s.t.} & \quad g_i(X) \leq b_i \forall i, X \in [0, 1].
\end{align*}
\]

(4)

where \(X\alpha = \{x | \mu_{\alpha}(x) \geq \alpha, \forall X \geq 0\}\), for each \(\alpha \in [0, 1]\).

This is the fundamental concepts of \(\alpha\)-level cuts method of fuzzy mathematical programming. The membership function in Eq. (3) indicates that if \(g_i(X) \in (b_i,b_i + p_i)\); then the memberships functions are monotonically decreasing. That also can means, the more resource consumed, the less satisfaction the decision maker thinks. One can then substitute Eq. (3) into Eq. (4) and obtain the following formulation:

\[
\begin{align*}
\min & \quad f(X) \\
\text{s.t.} & \quad g_i(X) \leq b_i + (1-\alpha) p_i \forall i
\end{align*}
\]

(5)

where \(X^L \leq X \leq X^U\) and \(\alpha \in [0, 1]\). Thus, the problem given in Eq. (5) is equivalent to a crisp parametric programming formulation while \(\alpha = 10\). For each \(\alpha\), one will have an optimal solution; therefore, the solution with \(\alpha\) grade of membership function is fuzzy. This model was applied by Wang et al. \[6\] and Rao \[7\] in structural design problems.

Werner’s Approach: Max-\(\alpha\) Method

Werner’s \[14\] proposed the objective function of Eq. (1) should be fuzzy due to the fuzziness existing in fuzzy inequality constraints. For solving Eqs. (1) to (2), one needs to define \(f_{\max}\) and \(f_{\min}\) as follows:

\[
\begin{align*}
f_{\max} & = \min f(X), \text{s.t. } g_i(X) \leq b_i \forall i, \text{ and } X^L \leq X \leq X^U \quad (6) \\
f_{\min} & = \min f(X), \text{s.t. } g_i(X) \leq b_i + p_i \forall i, \text{ and } X^L \leq X \leq X^U \quad (7)
\end{align*}
\]

The membership function \(m_f(X)\), as shown on Figure 2, of the objective function is stated as:

\[
m_f(X) = \begin{cases} 
1 & \text{if } f(X) < f_{\min} \\
1 - \frac{f(X) - f_{\min}}{f_{\max} - f_{\min}} & \text{if } f_{\min} \leq f(X) \leq f_{\max} \\
0 & \text{if } f(X) > f_{\max}
\end{cases}
\]

(8)

One can consequently apply the max-min operator to obtain the optimal decision. Then, Eqs. (1) and (2) can be solved by the strategy of max-\(\alpha\), where \(\alpha = \min[\mu_f(X), \mu_{g1}(X), \mu_{g2}(X), \ldots, \mu_{gm}(X)]\). That is:

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \alpha \leq \mu_f(X) \quad (9) \\
& \quad \alpha \leq \mu_{g_i}(X) \forall i \quad (10)
\end{align*}
\]

(11)

where \(\alpha \in [0, 1]\) and \(X^L \leq X \leq X^U\). This model is similar to the model proposed by Zimmermann \[15\] and applied in structural design by Rao \[8,9\].

Xu’s Approach: Bound Search Method

Suppose there are a fuzzy goal function \(f\) and a fuzzy constraint \(C\) in a decision space \(X\), which are characterized...
by their membership functions \( \mu_f(X) \) and \( \mu_C(X) \), respectively. The combined effect of those two can be represented by the intersection of the membership functions, as shown in Figure 3 and the following formulation.

\[
\mu_D(X) = \mu_f(X) \land \mu_C(X)
\]

(12)

Then Bellman and Zadeh [16] proposed that a maximum decision could be defined as:

\[
\mu_D(X^*) = \max \mu_D(X)
\]

(13)

If \( \mu_D(X) \) has a unique maximum at \( X^* \), then the maximizing decision is a uniquely defined crisp decision. From Eq. (13) and following the procedure given in [16], one can obtain the particular optimum level \( \alpha^* \) corresponding to the optimum point \( X^* \) such that:

\[
\mu(C^*) = \max_{X \in \alpha^*} \mu_C(X)
\]

(14)

where \( C^* \) is the fuzzy constraint set \( C \) of \( \alpha^* \)-level cut.

Xu [11] used a goal membership function of \( f(X) \) as following:

\[
\mu_{\alpha}(X) = \frac{f_{\min}}{f(X)}
\]

(15)

where \( f_{\min} \) has been defined in Eq. (7). It is apparent that the upper and lower bound of this goal membership function is between 1 and \( f_{\min}/f_{\max} \). One can apply Eq. (15) for Eq. (10); and as a result, the optimum \( \alpha^* \) can be achieved through an iteration computation. This method has been called the 2nd phase of \( \alpha \)-cuts method in Xu’s paper [11].

### 3. Level-Cut Approach of the First Kind for FNLP Problems

In Verdegay’s original \( \alpha \)-cuts method, each \( \alpha \) value can yield to an optimum solution \( X_\alpha \). Each constraint function in Eq. (5) using the same level of \( \alpha \) that makes this method as the single level-cut approach. Let us define \( f_{\max} - f_{\min} = p_f \), and \( f_{\min} = b_f \), then one can substitute Eq. (8) into Eq. (10) and substitute Eq. (3) into Eq. (11), the formulations of Eqs. (9–11) yield to:

Max \( \alpha \)

s.t. \( f(X) \leq b_f + (1 - \alpha) p_f \)

\( g_i(X) \leq b_i + (1 - \alpha) p_i, \forall i \)

(16)

(17)

where \( \alpha \in [0, 1] \) and \( X^d \leq X \leq X^u \). Obviously, Eq. (17) is the same as Eq. (5). Thus, Eqs. (9) and (16–17) construct a crisp NLP formulation and a unique optimum solution can be obtained. We called this formulation as the single level-cut approach of the first kind.

### 4. Level-Cut Approach of the Second Kind for FNLP Problems

It is observed in Xu’s approach that in Eq. (14) (Figure 3) where maximizing \( \mu_C(X) \) is similar to maximizing \( \alpha \) (Eq. 9) in Werner’s approach; therefore, one predicts the final result of those two approaches have the similar tendency, even though the form of their membership function is not the same, in which Werner’s approach uses the linear function (Eq. 8) and Xu’s approach uses the nonlinear function (Eq. 15).

For obtaining the unique solution of the original \( \alpha \)-level cuts approach in NLP problem with fuzzy resources as discussed in above paragraphs, we propose another alternative single level-cut approach called the single level-cut approach of the second kind. This approach contains both linear membership function (Eq. 18) shown in Figure 2 and nonlinear membership function (Eq. 15) of objective function shown in Figure 4.
The mathematical formulation of the fuzzy problem given in Eqs. (1–2) with unique α-cut level can be written in the following:

Find \( X, [\alpha] \)

\[
\begin{align*}
\text{Min } & f(X) \\
\text{s.t. } & g_i(X) \leq b_i, \forall i \quad \text{(for linear } \alpha_i(X)) \quad (19) \\
& g_i(X) \leq b_i + (1 - \alpha) p_i, \forall i \quad \text{(for nonlinear } \alpha_i(X)) \quad (20) \\
& \alpha \in [0, 1] \quad \text{(for linear } \alpha(X)) \quad (21) \\
& \alpha \in [\alpha_{min}, 1] \quad \text{(for nonlinear } \mu_i(X)) \quad (23)
\end{align*}
\]

where \( X^L \leq X \leq X^U \) and \( \mu_i(X) \) can be nonlinear (Eq. 15) or linear (Eq. 18) membership functions.

Algorithm. The solution procedure of single level-cut approach of the second kind involving the optimization of sub-problems to find \( f_{max} \) and \( f_{min} \) can be described as the following steps:

Step 1. Construct a nonlinear fuzzy constraints problem, as shown on Eqs. (1) and (5).

Step 2. Find \( f_{max} \) by minimizing \( f(X) \), s.t. \( g_i(X) \leq b_i, \forall i \), and \( X^L \leq X \leq X^U \).

Step 3. Find \( f_{min} \) by minimizing \( f(X) \), s.t. \( g_i(X) \leq b_i + p_i, \forall i \) and \( X^L \leq X \leq X^U \).

Step 4. Select linear \( \mu_i(X) \) (Eq. 18) or nonlinear \( \mu_i(X) \) (Eq. 15).

Step 5. Let \( \alpha \) is an additional design variable, where \( 0 \leq \alpha \leq 1 \) for linear \( \mu_i(X) \) and \( f_{min}/f_{max} \leq \alpha \leq 1 \) for nonlinear \( \mu_i(X) \).

Step 6. Construct the crisp nonlinear mathematical formulation of Eqs. (1) and (19–23), then one can solve it by the reliable nonlinear programming software.

5. Illustrative Engineering Design Examples

Two planar structural design examples illustrated the presenting approaches of the first and second kind in detail. A rather complicated and large-scale space structural design sustaining two load conditions is utilized for illustrating the single level-cut approach of fuzzy problem which also is compared with crisp design problem.

Example 1. Three-bar Planar Truss Design

The three-bar truss shown in Figure 5 where \( L = 1 \) m and \( p = 1000 \) N that is a common structural problem used by researchers in literatures to demonstrate the development of the optimum design algorithm. The same structure is also used here to illustrate the application of α-cuts methods in a nonlinear programming problem subjected to fuzzy resources. The cross-sectional areas of member 1 and 2, denoted as \( x_1 \) and \( x_2 \), are selected as design variables. The weight of the truss is minimized while the horizontal (\( u(X) \)) and vertical displacements (\( v(X) \)) of the loaded joint, the tensile stresses (\( \sigma_{1con}(X) \)) and compressive stresses (\( \sigma_{2con}(X) \)) in members are computed by finite element analysis that are taken as the design constraints. The Young’s modulus \( E \) and material density \( \rho \) are \( 2.05 \times 10^{11} \) Pa and \( 7.86 \times 10^3 \) kg/m³, respectively.

The linear fuzzy allowable ranges in fuzzy constraints are described in the following formulation that is adopted from the reference [18]. The allowable horizontal displacement of loading joint is \( 7.5 \times 10^{-6} \) m with a fuzzy region of \( 5.0 \times 10^{-6} \) m. The allowable vertical displacement of loading joint is \( 5.0 \times 10^{-6} \) m with a fuzzy region of \( 2.5 \times 10^{-6} \) m. The allowable tensile stress for both rod 1 and rod 2 is \( 1.25 \times 10^6 \) Pa with a fuzzy region of \( 5.0 \times 10^3 \) Pa. The allowable compressive stress in rod 3 is one-tenth of Euler buckling stress (\( \sigma_{Euler} \)) with a fuzzy region of \( 0.9 \times \sigma_{Euler} \). Consequently, the fuzzy mathematical formulation of single level-cut approach described in this paper is stated as:

\[
\begin{align*}
\text{Find } X &= [A_1, A_2, \alpha]^T = [x_1, x_2, x_3]^T \\
\text{Min } & f(X) = 2\sqrt{2} p x_1 + p x_2 \\
\text{s.t. } & g_1(X): \ u(X) \leq 7.5 \times 10^{-6} + 5 \times 10^{-6} (1 - \alpha) \quad (25) \\
& g_2(X): \ v(X) \leq 5 \times 10^{-6} + 2.5 \times 10^{-6} (1 - \alpha) \quad (26) \\
& g_3(X): \ \sigma_{1con}(X) \leq 1.25 \times 10^6 + 5 \times 10^3 (1 - \alpha) \quad (Pa) \quad (27) \\
& g_4(X): \ \sigma_{2con}(X) \leq 1.25 \times 10^6 + 5 \times 10^6 (1 - \alpha) \quad (Pa) \quad (28) \\
& g_5(X): \ \sigma_{3con}/\sigma_{Euler} (X) \leq 0.1 + 0.9 (1 - \alpha) \quad (Pa) \quad (29)
\end{align*}
\]
\begin{align*}
    f(X) &= [23.5 - \alpha(23.5 - 16.77)] = 0 \quad \text{(for linear } \mu(X)) \quad (30) \\
    f(X) &= (16.77/\alpha) \quad \text{(for nonlinear } \mu(X)) \quad (31)
\end{align*}

where

\begin{equation}
    \sigma_{\text{failure}}(X) = \pi E \sigma_x / 8
\end{equation}

and $10^{-4} \leq x_i \leq 10^{-2}, i = 1, 2 \text{ (m}^2\text{)}$. This problem was first solved by conventional $\alpha$-cuts method with fuzzy constraint representation (Eq. 5). It is noticed that the result of $\alpha = 1$ indicates the equivalent result of crisp optimum design. The value of $f_{\min}$ and $f_{\max}$ mentioned in the paper are obtained as 16.77 and 23.5 kg, respectively. Table 1 shows the optimum design of $x_1$ and $x_2$ corresponding to different cutting level of $\alpha$ using conventional $\alpha$-cuts approach solved by SQP in this paper and recorded in Ref. [18]. Accordingly, the optimum result using level-cut approach of the second kind is listed in Table 2.

Example 2. Ten-bar Planar Truss Design

A ten-bar truss problem is shown in Figure 6 [19]. The objective function is the total weight of the structure. The design variables are the cross-sectional areas of the 10 members with 0.1 in$^2$ as the minimum value. The constraints are the member stresses and the vertically nodal displacements where the nodes 2 and 4 sus-
tains vertical load $p$ of 100 kips. The allowable stress of each member is limited to $\pm 25$ ksi. The design parameters are: Young’s modulus $E = 10^4$ ksi and material density $\rho = 0.1$ lb/in$^3$. The linear fuzzy allowable range is $\pm 5$ ksi in constraints. Consequently, the fuzzy mathematical formulation of single level-cut approach described in this paper can be stated as: Find $X = [A_{11} A_{12}, \ldots, A_{10}, \alpha]^T = [x_1, x_2, \ldots, x_{10}]^T$

\begin{table}[h]
\centering
\caption{Optimum results of conventional level-cut approach for 3-bar truss design}
\begin{tabular}{|c|c|c|c|}
\hline
$\alpha$ & $x_1 \times 10^{-4}$ m$^2$ & $x_2 \times 10^{-4}$ m$^2$ & $f(X^*)$ \\
\hline
0 & 6.33 & 3.44 & 16.77 \\
0.1 & 6.51 & 3.54 & 17.26 \\
0.2 & 6.71 & 3.64 & 17.78 \\
0.3 & 6.92 & 3.75 & 18.34 \\
0.4 & 7.15 & 3.86 & 18.93 \\
0.5 & 7.39 & 3.99 & 19.56 \\
0.6 & 7.64 & 4.12 & 20.23 \\
0.7 & 7.91 & 4.27 & 20.96 \\
0.8 & 8.21 & 4.42 & 21.73 \\
0.9 & 8.62 & 4.32 & 22.57 \\
1.0 & 9.20 & 3.89 & 23.50 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Optimum results of single level-cut approach of the second kind for 3-bar design}
\begin{tabular}{|l|c|c|c|c|c|c|c|}
\hline
$\alpha$ & Linear $\mu(X)$ & $f(X^*)$ & $x_1^*$ & $x_2^*$ & $f_1^*$ & $f_2^*$ & $f_3^*$ & $f_4^*$ \\
\hline
0.1 & 19.843 & (7.494,4.049) & 0.543 & 0.543 & 0.659 & 0.913 & 0.543 & 1.0 & 1.0 \\
0.2 & 21.554 & (8.145,4.384) & 0.778 & 0.778 & 0.806 & 1.0 & 0.778 & 1.0 & 1.0 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Optimum results of conventional level-cut approach for 10-bar truss design}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
$\alpha$ & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0 \\
\hline
$f(X)$ & 1351.554 & 1374.698 & 1398.655 & 1423.466 & 1449.181 & 1475.844 & 1503.520 & 1532.257 & 1562.120 & 1593.178 \\
\hline
\end{tabular}
\end{table}

Figure 6. Ten-bar truss with loading.
Table 4. Optimum results of Ref. [19], using SQP of crisp problem and single level-cut approach of the second kind for 10-bar truss design

<table>
<thead>
<tr>
<th>Design variables</th>
<th>Ref. [19]</th>
<th>SQP</th>
<th>Level-cut approach (linear μf)</th>
<th>Level-cut approach (nonlinear μf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁</td>
<td>7.9379</td>
<td>7.9379</td>
<td>7.2408</td>
<td>7.7167</td>
</tr>
<tr>
<td>x₂</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>x₃</td>
<td>8.0621</td>
<td>8.0621</td>
<td>7.3651</td>
<td>7.8410</td>
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<td>x₄</td>
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<td>3.9379</td>
<td>3.5893</td>
<td>3.8273</td>
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<tr>
<td>x₇</td>
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<td>5.7447</td>
<td>5.2518</td>
<td>5.5884</td>
</tr>
<tr>
<td>x₈</td>
<td>5.5690</td>
<td>5.5690</td>
<td>5.0761</td>
<td>5.4126</td>
</tr>
<tr>
<td>x₉</td>
<td>5.5690</td>
<td>5.5690</td>
<td>5.0761</td>
<td>5.4126</td>
</tr>
<tr>
<td>x₁₀</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>α value</td>
<td></td>
<td></td>
<td>0.523</td>
<td>0.858</td>
</tr>
<tr>
<td>Optimum f(X) (lb)</td>
<td>1593.18</td>
<td>1593.18</td>
<td>1455.17</td>
<td>1549.40</td>
</tr>
</tbody>
</table>

Min f(X) = 360ρ (∑ᵢ₌₁⁹ xᵢ + √(2 ∑ᵢ₌₁⁹ xᵢ ))  

s.t. f(X) – [1593.18 – α(1593.18 – 1329.18)] = 0  

(α ∈ [0, 1] (for linear μf(X)))  

f(X) – (1329.18/α) = 0  (for linear μf(X))  

σᵢ(X) ≤ 25000 + 5000 (1 – α) (psi), i = 1, 2, ..., 10  

where α ∈ [0, 1] (for linear μf(X)) and α ∈ [1329.18/1593.18, 1] (for nonlinear μf(X)). The variable of α indicates the cutting level taken as an additional design variable between zero and one. Table 3 shows the optimum design corresponding to different cutting level of α using conventional α-cut approach. The values of 1329.18 and 1593.18 represent fₓₓ and fₓᵧ obtained by α equals to one and zero, respectively. Table 4 shows the final results of applying single level-cut approach of the second kind, the results obtained by SQP of crisp problem, and the original results [19] of the crisp problem. The last two columns represent the unique design obtained through linear and nonlinear objective membership function.

Example 3. Twenty-five-bar Space Truss Design

A 25-bar space truss shown in Figure 7 is required to support two load conditions given in Table 5 and is to be designed with constraints on member stresses as well as Euler buckling [17]. The allowable stress for all members is 40 ksi in both tension and compression. The Young’s modulus and the material density are taken as E = 10⁷ psi and ρ = 0.1 lb/in³, respectively. The members are assumed to be tubular with nominal diameter/thickness ratio of 100, so that the buckling stress in member i becomes:

Figure 7. Twenty-five-bar space truss.

Table 5. Load acting on the 25-bar truss [17]

<table>
<thead>
<tr>
<th>Load condition 1 (lbs)</th>
<th>Load condition 2 (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Joint 1</td>
</tr>
<tr>
<td>Fₓ</td>
<td>0</td>
</tr>
<tr>
<td>Fᵧ</td>
<td>20000</td>
</tr>
<tr>
<td>Fz</td>
<td>-5000</td>
</tr>
</tbody>
</table>
where $A_i$ and $l_i$ denote the cross-sectional area and length, respectively, of member $i$. The member area in this design is symmetrical arrangement so that they are linked as: $[A_1, A_2 = A_3, A_4 = A_5, A_6 = A_7, A_8 = A_9, A_{10} = A_{25}, A_{12} = A_{13}, A_{14} = A_{15}, A_{16} = A_{17}, A_{18} = A_{19}, A_{20} = A_{21}, A_{22} = A_{23} = A_{24} = A_{25}]^T = [x_1, x_2, \ldots, x_8]^T$. Each design variable corresponds to a member length of $l_i$. Thus there are eight independent area design variables with 100 design constraints. The original design formulations of minimizing structural weight can be written as: Find $X = [x_1, x_2, \ldots, x_8]^T$

$$\text{Min } f(X) = \rho \left( x_i l_i + 4x_i l_j + 4x_j l_i + 2x_i d_i + 2x_i d_j \right) + 4x_{d, i} + 4x_{d, j}$$

s.t. $|\sigma_{ij}(X) - 40000| \leq 0 + (\text{psi}),\quad i = 1, 2, \ldots, 25, j = 1, 2$

$$|\sigma_{ij}(X) - 40000 + 4000 (1 - \alpha) (\text{psi}),\quad i = 1, 2, \ldots, 25, j = 1, 2$$

(38) where $\sigma_{ij}$ is the stress induced in member $i$ under load condition $j$. $l_i (i = 1, 2, \ldots, 8)$ in the function of $f(X)$ represents the length of each member corresponding to each variable $x_i$. The bound of each design variable is written as $0.1 \text{ in}^2 \leq x_i \leq 5.0 \text{ in}^2$. The optimum results in Rao’s book [17] and SQP in present work are listed in the first and second column of Table 6. The objective value of 232.313 lb is expressed as $f_{\text{max}}$ in this paper. Then the problem was considered 20% fuzzy zone of allowable limit in constraints and the final objective value is 212.295 lb that is expressed as $f_{\text{min}}$. Consequently, one can apply single level-cut approach of the second kind described in this paper and fuzzy mathematical formulation can be stated as: Find $X = [x_1, x_2, \ldots, x_8]^T$ in order to minimize $f(X)$ that is the same as it in crisp formulation (38). The constrained functions are written as:

$$f(X) - [232.313 - \alpha(232.313 - 212.295)] =$$

(41)

$$f(X) - (212.295/\alpha) = 0 \text{ (for nonlinear } \mu(X))$$

(42)

$$|\sigma_{ij}(X) - 40000 + 8000 (1 - \alpha) (\text{psi}),\quad i = 1, 2, \ldots, 25, j = 1, 2$$

(43)

$$\sigma_{ij}(X) \leq \sigma_{ij}(X) + 0.2 \times \alpha \sigma_{ij}(X) (1 - \alpha) (\text{psi}),\quad i = 1, 2, \ldots, 25, j = 1, 2$$

(44)

where design variable $x_9$ represents level-cut value of $\alpha$. Its design bounds described as $x_9 \in [0, 1]$ (for linear $\mu(X)$) and $x_9 \in [212.295/232.313, 1]$ (for nonlinear $\mu(X)$). Thus, a fuzzy problem has been transformed to a crisp problem containing nine independent design variables with 101 design constraints in this crisp problem. The solution of this problem by the proposed approach of the second kind is shown on Table 6.

### 6. Discussions

From the previous article, one sees that the presenting single level-cut approach of the first kind formulated in Eqs. 9, 16 and 17 is inspiring from Rao’s and Zimmermann’s idea (Eqs. 9–11) [8, 9, 15]. Including linear $\mu(X)$

<table>
<thead>
<tr>
<th>Variables</th>
<th>Ref. [17]</th>
<th>SQP</th>
<th>Level-cut approach (linear $\mu$)</th>
<th>Level-cut approach (nonlinear $\mu$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.80228</td>
<td>0.80081</td>
<td>0.76513</td>
<td>0.79462</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.74789</td>
<td>0.74382</td>
<td>0.71007</td>
<td>0.73797</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0.12452</td>
<td>0.12425</td>
<td>0.11855</td>
<td>0.12326</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0.57117</td>
<td>0.56897</td>
<td>0.54361</td>
<td>0.56458</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0.97851</td>
<td>0.97362</td>
<td>0.92954</td>
<td>0.96597</td>
</tr>
<tr>
<td>$x_8$</td>
<td>0.80247</td>
<td>0.80269</td>
<td>0.76655</td>
<td>0.79643</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>value</td>
<td>0.51708</td>
<td>0.92095</td>
<td></td>
</tr>
<tr>
<td>Optimum weight (lb)</td>
<td>233.073</td>
<td>232.313</td>
<td>221.962</td>
<td>230.518</td>
</tr>
</tbody>
</table>

### Table 7. Optimum results of the first kind approach for 3-bar truss design

<table>
<thead>
<tr>
<th>$f(X)*$ kg</th>
<th>$(x_1^\ast, x_2^\ast) \times 10^4$ m$^2$</th>
<th>$\alpha^\ast$</th>
<th>$\alpha_{f1}$</th>
<th>$\alpha_{f2}$</th>
<th>$\alpha_{f3}$</th>
<th>$\alpha_{g1}$</th>
<th>$\alpha_{g2}$</th>
<th>$\alpha_{g3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear $\mu(X)$</td>
<td>19.844</td>
<td>(7.493, 4.051)</td>
<td>0.543</td>
<td>0.543</td>
<td>0.659</td>
<td>0.913</td>
<td>0.543</td>
<td>1.0</td>
</tr>
<tr>
<td>Nonlinear $\mu(X)$</td>
<td>21.628</td>
<td>(8.631, 3.105)</td>
<td>0.775</td>
<td>0.775</td>
<td>0.901</td>
<td>0.881</td>
<td>0.775</td>
<td>1.0</td>
</tr>
</tbody>
</table>
or nonlinear $\mu_\alpha(X)$, the complete mathematical formulation using the approach of the first kind for three-bar truss design is written as following:

Find $X = [x_1, x_2, x_3]^T = [A_1, A_2, \alpha]^T$

Max $\alpha$

s.t. $f(X) \leq 16.77 + (1 - \alpha) (23.5 - 16.77)$ (for linear $\mu_\alpha(X)$)

and other constraints of Eqs. (25–29), where $\alpha \in [0, 1]$ and $10^{-4} \leq x_i \leq 10^{-2}, i = 1, 2$ (m$^2$). Table 7 shows the optimum results using the first kind approach. As one compares Table 2 and Table 7, both results are very close with minor difference. When one analyzes both formulations, they have the same tendency of allowing achieve a unique design associated with a unique design level of $\alpha$. Until now, we can conclude that the presenting single level-cut approach of the second kind (Eqs. 1, 19–23) for fuzzy resources problems is the same as the approach of the first kind (Eqs. 9, 16, 17), so that both are recommended as alternative level-cut approaches for obtaining unique design. One can further investigate that the actual $\alpha$-value of constraints from $g_i(X)$ to $g_s(X)$ in Table 2 and Table 7 are also very close. The optimum level-cut value of $\alpha^*$ shown in Table 2 and Table 7 is the $\alpha$-value of objective function $\alpha_{\alpha^*}$, and also is the minimum one among $\alpha_{\alpha^*_i} (i = 1, 2, \ldots, 5)$ of all constraints. When one looks at ten-bar-truss design with the approach of the first kind, the optimum result listed in Table 8 that is very close to the result listed in Table 4. One can easily examine the actual $\alpha$-value of constraints from $g_i(X)$ to $g_s(X)$ in 10-bar truss problem are also very closed each other between the first and second kind of presenting single-cut approach.

### 7. Conclusions

Two alternative level-cut (alpha-cut) approaches, the first and second kind, for solving nonlinear design problems with fuzzy resources problems have been presented in this paper. The approach of the second kind, as formulated as equations (1) and (19–23), is the most recommended for its easy formulation with increasing only one equality constrained equation (19 or 20) and one additional design variable $\alpha$. The approach of the first kind is secondly recommended approach, as formulated in Eqs. 9, 16, and 17, which requires to increase additional inequality constrained function (Eq. 16) and one additional design variable $\alpha$ also. Additionally, the first kind approach requires a scalar objective function (Eq. 9) instead of the original objective function (Eq. 1). However, both kinds of approaches can yield to the same optimum result. Linear or nonlinear formulation of $\mu_\alpha(X)$ can be selected by designer depending on the shape of fuzzy region. The proposed two single level-cuts methods can guarantee to obtain the unique compromise final design.

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### References


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