Structurally Constrained $H_\infty$ Suboptimal Control Problems: Low-order Controller Design

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Abstract

This paper discusses a new iterative method for designing an $H_\infty$ low-order controller. The method is based on the optimization of a structurally constrained constant gain feedback controllers with linear matrix inequalities as the design constraints. In this new approach, the control gains are independent of any Riccati equation solutions. Controller constraints such as decentralization and positive realness can also be included in the design.

Key Words: Low-order controller, Linear matrix inequalities, $H_\infty$ optimization

1. Introduction

In the $H_\infty$ design theory [3], the order of the controller is the same as the order of the generalized plant including those of the weighting functions. In practical control designs, low-order controllers are usually desired for reliability and ease of implementation. There are several general approaches to design low-order stabilizing controllers. One of the approaches is first to reduce the order of the high-order plant, and then design a low-order controller based on the reduced plant model. A potential problem of this approach is that the closed-loop full-order plant with the low-order controller may be unstable because the model error is not considered in the design process. A second approach is first to design a high-order controller based on the full-order plant model. Then a low-order stabilizing controller can be obtained via controller reduction techniques [6,8,9,17]. The major considerations of the controller order reduction are to guarantee the closed-loop stability and to minimize the performance degradation. Perhaps the most desirable approach is to directly design a low-order controller for a high-order plant.

Recently, it has been shown that $H_\infty$ control optimization can be solved using linear matrix inequalities (LMI's) [1,13]. For design problems with controller structural constraints such as fixed controller order, it is well known that the design equation involves a biaffine matrix inequality (BMI) which is a non-convex programming problem and cannot be solved in polynomial time. As a result, several methods (such as the alternating projection method, the V-K iteration, and a rank condition minimization method) to reformulate the design problem into the solutions of recursive sets of LMI's have been proposed [6,7,8,9,10,13,14]. Although these methods do not have global convergence properties, they can converge to local solutions. However, these approaches do not seem to readily admit controller constraints such as a minimum phase condition.

The iterative LMI-based technique in [19] to design a structurally constrained state feedback controller allows a designer to impose certain desirable constraints on the state feedback gain matrix. In this paper, we will discuss a procedure for the direct design of $H_\infty$ low-order controllers based on the structurally constrained state-feedback formulation. Since the problem cannot be solved directly, the controller parametrization result in [3] is used to develop a dual design problem, which is solved iteratively as a sequence of LMI problems using [5]. In the proposed method, if the poles of the controller are pre-determined based on design considerations such as the controller bandwidth, then a minimum phase controller will be obtained.

Two design examples, the control of two inverted pendulums in cascade and the design of a power system stabilizer for a single-machine infinite-bus power system, are used to demonstrate the direct design of $H_\infty$ lower-order controller using the proposed method.
follows. Section 2 presents the general formulation of the $H_\infty$ low-order controller design. Section 3 discusses the fixed pole design problem. The solution algorithm is given in Section 4. Design examples are shown in Section 5.

2. General Formulation

Consider the linear time-invariant generalized plant $G(s)$ with the state-space realization

$$\begin{align*}
\dot{x} &= Ax + Bu + B_2 w + B_4 u \\
z &= C_1 x + D_1 w + D_2 u \\
y &= C_2 x + D_2 u
\end{align*}$$

where $x \in \mathbb{R}^n$ is the state variable vector, $w \in \mathbb{R}^m$ represents the disturbance and other external input vector, $z \in \mathbb{R}^q$ is the controlled output vector, $u \in \mathbb{R}^n$ is the controlled input vector, and $y \in \mathbb{R}^q$ is the measured output vector. The following assumptions are made for this control problem:

i) $(A, B_2)$ is stabilizable and $(A, C_2)$ is detectable

ii) $[A - j\omega B_2]$ has full column rank for all $\omega$

iii) $D_{21}$ is of full column rank

iv) $C_2$ is of full row rank

v) $D_{21} = 0, D_{22} = 0$

Without loss of generality, we assume that $C_2$ has the form

$$C_2 = \begin{bmatrix} I & 0 \end{bmatrix}$$

If this is not the case, we can perform a state transformation on $x$, such that (2) is satisfied. We further note that proper weighting functions/filters may be required to ensure that assumption (v) is satisfied.

The $H_\infty$ low-order controller design is to find a control $u = K(s)y$, as shown in Figure 1, such that the closed-loop transfer function from $w$ to $z$, denoted as $T_{zw}$, is stable and

$$\|T_{zw}\| < \gamma$$

where $\| \cdot \|$ is a pre-specified constant, and $K(s)$ has the state space realization

$$\begin{align*}
\dot{x} &= A_x x + B_1 y \\
u &= C_x x + D_1 y
\end{align*}$$

where $x \in \mathbb{R}^n$ is the state variable vector, and the controller order is specified as $n_c \leq n$. Finding an $H_\infty$ low-order controller (5) is a non-convex problem. There is no closed form solution to the problem. Finding a numerical solution for the problem is also difficult in general. To simplify the design, we reformulate the design problem into a more tractable situation.

The closed-loop system (1) subject to the low-order controller (5) is

$$\begin{align*}
\dot{x} &= Ax + Bu + B_2 w + B_4 u \\
z &= C_1 x + D_1 w + D_2 u \\
y &= C_2 x + D_2 u
\end{align*}$$

which is equivalent to the following augmented system

$$\begin{align*}
\dot{x} &= \bar{A} x + B_2 w + B_4 u \\
z &= \bar{C}_1 x + \bar{D}_{12} y \\
y &= C_2 x
\end{align*}$$

subject to the control

$$v = \begin{bmatrix} A_x & B_1 \\ C_x & D_1 \end{bmatrix} y$$

where

$$\begin{align*}
x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
y &= \begin{bmatrix} x_3 \\ y_1 \end{bmatrix} \\
\bar{A} &= \begin{bmatrix} 0 & 0 \\ 0 & A_y \end{bmatrix}, \\
\bar{B}_i &= \begin{bmatrix} 0 \\ B_{2i} \end{bmatrix}, \\
\bar{B}_i &= \begin{bmatrix} f \ 0 \\ 0 \ B_i \end{bmatrix} \\
\bar{C}_1 &= \begin{bmatrix} 0 & C_1 \\ 0 & C_2 \end{bmatrix}, \\
\bar{D}_{12} &= \begin{bmatrix} 0 & D_{12} \end{bmatrix}
\end{align*}$$

We note that in the augmented system (7), $\bar{D}_{12}$ is not of full column rank, and $[\bar{A} - j\omega \bar{B}_2 \bar{C}_1 \bar{D}_{12}]$ loses rank at $\omega = 0$. In order to use the result of structurally constrained state-feedback design [19], we let

$$A_x = A_{0x} + A_c$$

where $A_{0x}$ is a constant matrix, and fix $C_x$ to reduce...
the design variables. Then the design system (7) is translated into the following generalized system

\[ \dot{x} = Ax + B_1 w + B_2 y \]  
\[ z = C_1 x + D_{12} y \]  
\[ y = C_2 x \]  
subject to the control

\[ v = \begin{bmatrix} A_{wv} & B_k \\ 0 & D_k \end{bmatrix} y \]

where

\[ A = \begin{bmatrix} A_{0} & 0 \\ B_k C_k & A_y \end{bmatrix}, D_{12} = \begin{bmatrix} \alpha I & 0 \\ 0 & D_{g2} \end{bmatrix} \]

\[ C_1 = \begin{bmatrix} 0 & 0 \\ D_{g2} & C_{g2} \end{bmatrix}, \quad z = \begin{bmatrix} z_k \\ z \end{bmatrix} \]

such that the closed-loop transfer function from \( \omega \) to \( z \), denoted \( T_{2w} \), is stable and

\[ \| T_{2w} \|_{\infty} < \gamma \]  

We further note that if a strictly proper controller is selected, i.e., \( D_{g2} = 0 \), then the design system (13) subject to the control (18) can be reduced to the following generalized system

\[ \dot{x} = Ax + B_1 w + \begin{bmatrix} 0 \\ I \end{bmatrix} y \]  
\[ z = C_1 x + \begin{bmatrix} \alpha I \\ 0 \end{bmatrix} y \]  
\[ y = C_2 x \]
subject to the control

\[ v = \begin{bmatrix} A_{wv} & B_k \end{bmatrix} y = \begin{bmatrix} A_{wv} & B_k \\ 0 & 0 \end{bmatrix} x \]

\[ F_c x \]

where \( F_c = \begin{bmatrix} A_{wv} & B_k \\ 0 & D_k \end{bmatrix} \) is a structurally constrained state feedback gain matrix. Thus the low-order control design problem is reformulated as a structurally constrained state feedback control problem.

In this design, it is assumed that (1) can be stabilized by a controller of order \( n_c \). This assumption cannot be checked beforehand, unless one already has such a stabilizing controller. In practice, a designer would pick a controller order and proceed to compute a solution. If the design algorithm fails to give a solution, the order \( n_c \) is increased. It is important to note that the scalar \( \alpha \) introduced in \( D_{g2} \) implies that the state of the controller is penalized. Therefore, \( \alpha \) shall be selected such that the \( H_\infty \) norm of the closed-loop system from \( w \) to \( z_k \) does not dominate the design requirement.

We further note that if a strictly proper controller is selected, i.e., \( D_{g2} = 0 \), then the design system (13) subject to the control (18) can be reduced to the following generalized system

\[ \dot{x} = Ax + B_1 w + \begin{bmatrix} 0 \\ I \end{bmatrix} y \]  
\[ z = C_1 x + \begin{bmatrix} \alpha I \\ 0 \end{bmatrix} y \]  
\[ y = C_2 x \]
subject to the control

\[ v = \begin{bmatrix} A_{wv} & B_k \end{bmatrix} y = \begin{bmatrix} A_{wv} & B_k \\ 0 & 0 \end{bmatrix} x \]

\[ F_c x \]

where \( F_c = \begin{bmatrix} A_{wv} & B_k \\ 0 & D_k \end{bmatrix} \) is a structurally constrained state feedback gain matrix.

3. Fixed-pole Design

In many control systems, a designer may be able to specify the poles of the controller quite readily, based on the control bandwidth specifications. The model-matching approach introduced by Edmunds [4] provides a procedure for specifying the poles of the desired controller. With the poles fixed, only the gains and the zeros need to be optimized. Following the development in the last section, if the poles of the controller are determined a priori, then the low-order controller design for the generalized system (1) can be reformulated as finding a static output feedback control

\[ v = \begin{bmatrix} C_1 & D_k \end{bmatrix} y \]

for the generalized system

\[ \dot{x} = Ax + B_1 w + B_2 y \]  
\[ z = C_1 x + D_{12} y \]  
\[ y = C_2 x \]

where

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \]

\[ A = \begin{bmatrix} A_{0} & B_k C_{g2} \\ 0 & A_y \end{bmatrix}, B_k = \begin{bmatrix} 0 \\ B_{g2} \end{bmatrix}, \quad B_{g2} = \begin{bmatrix} 0 \\ B_{g2} \end{bmatrix} \]

\[ C_1 = \begin{bmatrix} 0 & C_{g2} \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} I \\ 0 \\ C_{g2} \end{bmatrix} \]

such that the closed-loop system is stable and the transfer function from \( w \) to \( z \), which is exactly \( T_{2w} \) satisfies (4). The fixed poles are used to determine \( A_k \) and \( B_k \) using a canonical representation [15]. In this case, the control
\[ v = [C_k \ D_k] y = [C_k \ D_k \ 0] x \]
\[ F_d = [C_k \ D_k \ 0] x \] (26)
again, \( F_d = [C_k \ D_k \ 0] x \) is a structurally constrained state feedback gain matrix.

One of the major concerns with state-space design methods is that, in general, it is not possible to directly impose constraints on the controller itself. As a result, the design may provide a controller having poles and zeros in the right half-plane, even though a stable minimum-phase controller exists. In the fixed-pole controller design, we can impose constraints such as positive realness and decentralized structure directly on the controller. For example, if it is desired that the controller be positive real, the LMI constraints \[ \begin{bmatrix} A_k^T P + PA_k & PB_k - C_k^T \\ B_k^T P - C_k & -(D_k + D_k^T) \end{bmatrix} \leq 0 \] (27)
\[ P = P^T > 0 \] (28)
can be added to the design formulation, where \( P \) is an \( n \times n \) positive definite matrix. With this constraints, the transfer function of the designed low-order controller will be positive real. Hence the controller will be minimum phase and has no right-half-plane zeros. This is an important design condition in many practical control systems. For example, in power system damping controller design, the minimum-phase lead-lag compensators have been shown to provide good performance. And, therefore, a positive real controller is highly desired.

### 4. Solution Algorithm

The \( H_\infty \) structurally constrained state feedback control problem of finding \( E_1 \) (18,20,26) for the generalized system (13,19, and 22 respectively) to satisfy (4) is a non-convex programming problem. There is no known closed-form solution to the problem. We approach this problem by formulating a dual design problem, resulting in an optimization problem with quadratic matrix inequality (QMI) constraints. The QMI optimization problem can then be solved iteratively as an LMI problem. Our approach is summarized in Theorems 1 and 2.

We assume that the generalized system (13,19, and 22) satisfies the following assumptions:

- **(A1)** Systems (13,19,22) are stabilizable under the control structure (18,20, and 26 respectively). 

\[ \begin{bmatrix} A - j \omega I & B_k \\ C_1 & D_{12} \end{bmatrix} \] has full column rank for all \( \omega \) (A3) \( D_{12} \) is of full column rank.

**Theorem 1** [19] Suppose the generalized plants (13,19,22) satisfy Assumptions (A1)-(A3) and \( D_{12} \) has the singular value decomposition \( D_{12} = U \Sigma V^T \), where \( U \) and \( V \) are unitary matrices and \( \Sigma \) is a diagonal matrix. Then, for a given \( \gamma > 0 \), if the algebraic Riccati equation
\[ A_k^T X_{\gamma} + X_{\gamma} A_k + X_{\gamma} R_k X_{\gamma} - Q_k = 0 \] (29)
admits a positive-definite matrix solution \( X_{\gamma} \), where
\[ A_{\gamma} = A - B_2 (D_{12}^T D_{12})^{-1} D_{12}^T C_1 \] (30)
\[ R_k = \gamma^2 B_k^T B_k - B_2 (D_{12}^T D_{12})^{-1} B_2^T \] (31)
\[ Q_k = -C_1^T (I - D_{12}^T D_{12})^{-1} D_{12}^T C_1 \] (32)
and the structurally constrained feedback matrix \( F_d \) (18,20,26) is chosen such that a dual system in the packed matrix form satisfy
\[ \begin{bmatrix} A_{\text{mp}} + B F_d \\ S_{\gamma}(F_d - F) \\ 0 \end{bmatrix} < 0 \] (33)
where
\[ A_{\text{mp}} = A + \gamma^2 B_k^T X_{\gamma} \] (34)
\[ S_{\gamma} = \Sigma V^T \] (35)
\[ F = -(D_{12}^T D_{12})^{-1} (B_2^T X_{\gamma} + D_{12}^T C_1) \] (36)
then the controller (5) is a stabilizing controller for generalized system (1) satisfying \( \| T_{xu} \|_\infty < \gamma \) (4)

In Theorem 1, the \( H_\infty \) design problem is to find the control (5) by optimizing (33) instead of \( \| T_{xu} \|_\infty < \gamma \) (4). Thus the optimization of (33) is called a dual design problem. The result of Theorem 1 arises from the parametrization of all controllers satisfying (4) is a dual design problem. The result of Theorem 1 arises from the parametrization of all controllers satisfying (4) is a QMI problem which is, in general, difficult to solve. However, we introduce a free matrix parameter X to form a second QMI problem which can be solved as a sequence of LMI problems. The required design equations are stated in the following theorem.

**Theorem 2** [19] If there exist \( M \geq 0 \), \( F_d \) having the desired structure, and \( X \) such that the QMI
for

S4: Solve the following optimization problem OP

\[
\begin{bmatrix}
\Delta & \gamma^{-1} MB_i & \Phi^T S_u^T \\
\gamma^{-1} B_j^T M & -I & 0 \\
S_u \Phi & 0 & -I
\end{bmatrix} \leq 0
\] (37)

where

\[
\Delta = A_f^T M + M A_f - X^T B_u M
\]

\[- M B_u X + X^T B_u X
\]

\[
\Phi = F_d - F + S_u^T S_u^T B_2^T M
\]

(38)

(39)

(40)

then the control (5) satisfies \( \Box \) T \( \Box \) \( \Box \) (4).

The QMI (37) points to an iterative approach to solve for \( F_d \), namely, if \( X \) is fixed in (38), then (37) reduces to an LMI problem for a given \( \Box \) in the unknowns \( F_d \) and \( M \geq 0 \). The LMI problem is convex and can be solved, if a feasible solution exists, using existing LMI solvers [5] which use efficient interior point solution techniques. An iterative design procedure summarized below is introduced in [19] in which at every iteration, the trace of \( M \) is minimized. Then \( X \) is set to \( M \) and the structure of \( F_d \) is updated for the next iteration. When the iterative algorithm converges, the controller parameters can be obtained from \( F_d \). This iterative method does not have global convergence, but will converge if the initial full-state feedback solution \( F \) is close to a local solution. In addition, if the method fails to converge, it does not necessarily mean that there is no solution.

**Iterative LMI (ILMI) Algorithm**

S1: Select \( \gamma > 0 \) and compute the centralized solutions \( X_0 \) and \( F \). Separate the gains of \( F \) conforming to the structure of \( F_d \) and put them in \( F_d \). Write \( F \) as

\[
F = F_d + F_{con}
\]

where \( F_{con} \) contains all the other gains.

S2: Set up the sequence \( \Box \) as

\[
\epsilon_k = 1 - \frac{k}{N}, \quad k = 1, 2, \cdots, N
\]

(42)

S3: Set \( k = 1 \) and \( X^{(1)} = X_0 \).

S4: Solve the following optimization problem OP for \( M^{(k)} \) and \( F^{(k)} \).

OP: Minimize \( \text{trace}(M^{(k)}) \) subject to the following LMI constraints

\[
\begin{bmatrix}
\Delta^{(k)} & \gamma^{-1} M^{(k)} B_i & \Phi^{(k)} S_u^T \\
\gamma^{-1} B_j^T M^{(k)} & -I & 0 \\
S_u \Phi^{(k)} & 0 & -I
\end{bmatrix} \leq 0
\]

(43)

\[
M^{(k)} = (M^{(k)})^T \geq 0
\]

where

\[
\Delta^{(k)} = A_f^T M^{(k)} + M^{(k)} A_f - X^{(k)^T} B_u M^{(k)}
\]

\[- M^{(k)} B_u X^{(k)} + X^{(k)^T} B_u X^{(k)}
\]

\[
\Phi^{(k)} = F_d^{(k)} + F_{con} - F + S_u^T S_u^T B_2^T M^{(k)}
\]

(45)

(46)

S5: If \( \Box \) \( \Box \) \( \Box \), a pre-determined tolerance, go to Step S6. Else set \( X^{(k+1)} = X^{(k)} \) and go to Step S4.

S6: If \( k = N \), stop. Else set \( X^{(k+1)} = X^{(k)} \) and increment \( k \) to \( k + 1 \). Go to Step S4.

In the fixed-pole controller design, because \( A_k \) and \( B_k \) are already selected, (27) and (28) are linear in the variables \( C_k, D_k, \) and \( P \). Then the design parameters \( M, F_d = [C, D, 0] \), \( X \), and \( P \) need to be selected to satisfy the inequalities (37), (27), and (28). The resulting low-order controller will then be minimum phase and has no right-half-plane zeros.

**5. Design Examples**

In this section, two examples are selected to illustrate the design of \( H \_9 \)-low-order controllers. The first example is the benchmark problem in [2] of controlling two inverted pendulums in cascade. The second example is the design of a power system stabilizer (PSS) for a single-machine infinite-bus power system.

**Example 5.1**. The inverted pendulums model expressed in (1) has the state-space matrices [2].

\[
A_k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.8 & 0 & -9.8 & 0 \\ 0 & 0 & 0 & 1 \\ -9.8 & 0 & 2.94 & 0 \end{bmatrix}, \quad B_{g1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
B_{g2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \quad C_{g1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_{g2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
D_{g12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

The first two states are the angular displacement and the velocity of the first pendulum, and the last two states are the angular displacement and the velocity of the second pendulum. In this example, two examples are selected to illustrate the design of \( H \_9 \)-low-order controllers. The first example is the benchmark problem in [2] of controlling two inverted pendulums in cascade. The second example is the design of a power system stabilizer (PSS) for a single-machine infinite-bus power system.
example, we assume that only the angular displacements of the pendulums are available for feedback. The system cannot be stabilized via static output feedback. Hence a dynamic controller need to be designed. The order of the system is four. We will directly design a second-order $H_\infty$ controller. In this design, a strictly proper controller is considered. We first set $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Box = 1$, then select $A_{k0}$ as $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Using the structurally constrained state feedback design formulation and the IMLI algorithm, we obtain, for $\Box = 10$, the low-order $H_\infty$ controller

$$K(s) = \begin{bmatrix} 0.8727 & 17.847 & -59.901 & 43.174 \\ -5.8978 & -18.972 & 66.202 & -48.867 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (47)$$

The $H_\infty$ norm of the resulting closed-loop system is found to be 9.1785.

Example 5.2. In this example, we will directly design an $H_\infty$ second-order power system stabilizer (PSS) for a single-machine infinite-bus power system. The design model used for this PSS design is represented by an isolated generating unit connected to a bulk power system modeled by an infinite bus through a transmission line as shown in Figure 2. The dominant mode of the system is the oscillation of the machine versus the infinite bus. The PSS is designed with machine speed as the input signal. The stabilizer provides a supplementary signal to the exciter to modulate the generator field voltage in order to damp the generator oscillation.

The major uncertainty of the system is the variation of the equivalent impedance of the transmission line connecting the generator to the infinite bus. Three operating conditions are specified, representing the median, stiff, and weak transmission systems according to the impedance of the transmission line. The median system represents the nominal operating condition. When the impedance is low, the connection to the infinite bus is stiff and the damping ratio of the open-loop system is high. When the impedance is high, the connection to the infinite bus is weak and the damping ratio of the open-loop system is low and may even be negative (unstable). The linearized state space models of the median, stiff, and weak systems can be found in [20].

The system uncertainty modeled to represent the transmission strength variations is given by the perturbed input-output relationship [21]

$$\delta I = \delta Y \delta V \quad (48)$$

where $\Box Y$ is the admittance variations of the transmission line, $\Box I$ is the perturbed equivalent current injection on Buses 2 and 3, and $\Box V = V_2 - V_3$ is the voltage difference across the transmission line 2-3.

The purpose of the PSS design in this example is to design a low-order controller such that the closed-loop transfer function from $\Box I$ to $\Box V$ is stable and the $H_\infty$ norm of the closed-loop transfer function is less than a pre-determined bound $\Box$. The order of the linearized power system model is 8. In this example, the following low-pass filter is added at the external input terminal

$$W_p = \frac{100}{s + 100} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (49)$$

and the following washout and torsional filter are added at the controlled input

$$\text{Washout: } \frac{s}{s + 0.1} \quad (50)$$

$$\text{Torsional-filter: } \frac{1}{0.0017 s^2 + 0.061 s + 1} \quad (51)$$

Therefore, the total order of the generalized plant used for the design is 13. A second-order $H_\infty$ controller will be designed to improve the damping of all three operating conditions.

In this design, the poles of the controller are set at $-22$ with a multiplicity of 2. We further restrict the controller to be positive real. Using the design procedure discussed in the previous section, we obtain, for $\Box = 2$, low-order $H_\infty$ controller

$$K(s) = \frac{116.02(s + 7.2235)(s + 1.1186)}{(s + 22)^2} \quad (52)$$

The frequency response of the low-order controller (52) is shown in Figure 3. The damping ratios for
the three operating conditions are summarized in Table 1.

Table 1: Damping Ratio (%) of the single-Machine Infinite Bus System

<table>
<thead>
<tr>
<th>System</th>
<th>median</th>
<th>stiff</th>
<th>week</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open-loop</td>
<td>0.83</td>
<td>4.35</td>
<td>-2.75</td>
</tr>
<tr>
<td>Closed-loop</td>
<td>20.53</td>
<td>15.17</td>
<td>14.77</td>
</tr>
</tbody>
</table>

Figure 3. Frequency Response of the low-Order Controller $K(s)$

We note that in the open-loop system, the weak system is unstable while the median system is lightly damped. We also note that the median system is used in the design process. The designed low-order $H_{\infty}$ controller not only enhances the damping of the median system but also provides good damping for both the stiff and weak systems. In addition, the controller is minimum phase, that is, the controller has no right-half-plane poles or zeros, and thus would not interfere with other control functions such as transient stabilization.

6. Conclusion

An iterative LMI algorithm has been proposed for the design of an $H_{\infty}$ low-order controller. The design problem is reformulated as structurally constrained state-feedback $H_{\infty}$ suboptimal control problem. The technique is based on optimizing with respect to a dual design formulation, rather than optimizing directly the closed-loop system. The structurally constrained state feedback gain obtained from the iterative LMI algorithm does not depend on the positive definite Riccati solution from the design conditions. If poles and zeros of the controller are included in the search, then the state of the controller has to be penalized. On the other hand, if the poles of the low-order controller are pre-specified, then the low-order design can be directly formulated as a structurally constrained state-feedback control problem. Moreover, the positive real constraint may be imposed into the design process to obtain a minimum phase controller.

The technique has been successfully applied to the design of controlling two inverted pendulums in cascade and the design of a power system stabilizer for a single-machine infinite-bus power system.

Reference


